Let $!n = \#\{\sigma : \underset{\{0,1,\ldots,n-1\}}{\underset{\|}{n}} \to n : \sigma(x) \neq x \quad \forall x = 0,1,\ldots,n-1\}$

I use following 3 identities in this proof ;

$$① \quad !(n+2) = (n+1) \cdot \{!(n+1) + !n\}$$

$$② \quad !(n+3) = (n+3) \cdot !(n+2) + (-1)^{n+3}$$

$$③ \quad \lim_{n \to \infty} \frac{n!}{!n} = e$$

Not only $(n+1)a_n$ and $!(n+2)$ has the same recurrence relation, but also $1 \cdot a_0 = !2 = 1$ and $2 \cdot a_1 = !3 = 2$. Therefore $(n+1)a_n = !(n+2)$.

Using this, the series can be rewritten as

$$\sum_{n=0}^{\infty} (-1)^n \frac{n!}{a_n a_{n+1}} = \sum_{n=0}^{\infty} (-1)^n \frac{(n+2)!}{!(n+2)\,!(n+3)} .$$

Because of ②, this can be changed into telescoping sum as follows :

$$\sum_{n=0}^{\infty} (-1)^n \frac{(n+2)!}{!(n+2)\,!(n+3)} = \sum_{n=0}^{\infty} (n+2)! \left\{ \frac{n+3}{!(n+3)} - \frac{1}{!(n+2)} \right\}$$

$$= \sum_{n=0}^{\infty} \left(\frac{(n+3)!}{!(n+3)} - \frac{(n+2)!}{!(n+2)} \right)$$

$$= -\frac{2!}{!2} + \lim_{n\to\infty} \frac{(n+3)!}{!(n+3)}$$

$$= -2 + e.$$