KAIST POW 2017-13 : Infinite series with recurrence relation

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We define additional sequence b_n as follows: $b_0 = 1$, $b_1 = 2$, and $b_n = (n+1)b_{n-1} + nb_{n-2}$ for $n \geq 2$. Then we have following lemma.

Lemma 1. For each $k \geq 0$,

$$\sum_{n=0}^{k} (-1)^n \frac{n!}{a_n a_{n+1}} = \frac{b_k}{a_{k+1}}$$

Proof. Trivial if $k = 0$, Then it is enough to show that for each $k \geq 1$,

$$(-1)^k \frac{k!}{a_k a_{k+1}} = \frac{b_k}{a_{k+1}} - \frac{b_{k-1}}{a_k}$$

, which is equivalent to

$$(-1)^k k! = b_k a_k - b_{k-1} a_{k+1}$$

for each $k \geq 1$. It is trivial if $k = 1$, and we may use mathematical induction since

$$\begin{aligned}
&b_{k+1}a_{k+1} - b_k a_{k+2} \\
&= ((k+2)b_k + (k+1)b_{k-1})a_{k+1} - b_k((k+2)a_{k+1} + (k+1)a_k) \\
&= -(k+1)(b_k a_k - b_{k+1} a_{k+1})
\end{aligned}$$

□

Hence, the problem is equivalent to check limit of the sequence b_k/a_{k+1} : To adjust index, we may write $c_k = a_{k+1}$. Then it satisfies $c_0 = 1, c_1 = 3$ and $c_n = (n+1)c_{n-1} + nc_{n-2}$, which is same recurrence formula with b_n. Then for each $n \geq 2$, we have the formula

$$\frac{b_n}{c_n} = \frac{(n+1)b_{n-1} + nb_{n-2}}{(n+1)c_{n-1} + nc_{n-2}}$$

and we can show that b_n/c_n becomes convergent of generalized continued fraction [Wikipedia, "generalized continued fraction"]

$$\cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \cfrac{4}{5 + \cdots}}}}}$$

, which can be trivially modified as (just multipling from above)

$$\cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \cfrac{5}{5 + \cfrac{6}{6 + \cdots}}}}}$$

To calculate this continued fraction, apply Euler's continued fraction formula[Wikipedia, "Euler's continued fraction formula"] for $e^{-1} = 1 - 1 + 1/2! - 1/3! + 1/4! - \cdots$ becomes

$$e^{-1} = \cfrac{1}{1 - \cfrac{-1}{0 - \cfrac{-1}{1 - \cfrac{-2}{2 - \cfrac{-3}{3 + \cdots}}}}} = \cfrac{1}{1 + 1 + \cfrac{2}{2 + \cfrac{3}{3 + \cdots}}}$$

Then taking inverse and deleting 2 yields the answer :

$$\cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \cfrac{5}{5 + \cfrac{6}{6 + \cdots}}}}} = e - 2.$$