POW: 2017-11 Infinite series

2014[redacted] Jo Tae Hyouk

May 30, 2017

Let $H(n) = \sum_{k=1}^{n} \frac{1}{k}$ and $S = \sum_{n=1}^{\infty} H(n) \frac{1}{n(2n-1)}$. Note that for $0 < x < 1$,

$$-\frac{\log(1-x)}{1-x} = \sum_{k=0}^{\infty} x^k \sum_{n=1}^{\infty} \frac{x^n}{n} = \sum_{n=1}^{\infty} H(n) x^n.$$

Thus

$$\begin{aligned}
\frac{S}{2} &= \sum_{n=1}^{\infty} H(n) \left(\frac{1}{2n-1} - \frac{1}{2n} \right) \\
&= \int_0^1 \sum_{n=1}^{\infty} H(n)(x^{2n-2} - x^{2n-1}) dx \\
&= \int_0^1 \left(-\frac{\log(1-x^2)}{x^2(1-x^2)} + \frac{\log(1-x^2)}{x(1-x^2)} \right) dx \\
&= \int_0^1 -\frac{\log(1-x^2)}{x^2(1+x)} dx.
\end{aligned}$$

Write

$$\begin{aligned}
-\frac{\log(1-x^2)}{x^2(1+x)} &= -\frac{\log(1-x^2)}{x} \left(\frac{1}{x} - \frac{1}{1+x} \right) \\
&= -\frac{\log(1-x^2)}{x^2} + \frac{\log(1-x^2)}{x(1+x)} \\
&= -\frac{\log(1-x^2)}{x^2} + \frac{\log(1-x^2)}{x} - \frac{\log(1-x^2)}{1+x} \\
&= -\frac{\log(1-x^2)}{x^2} + \frac{\log(1-x^2)}{x} - \frac{\log(1-x)}{1+x} - \frac{\log(1+x)}{1+x}.
\end{aligned}$$

The integral of the first term can be calculated by using integration by parts:

$$\begin{aligned}
\int -\frac{\log(1-x^2)}{x^2} dx &= \frac{\log(1-x^2)}{x} - \int \frac{1}{x} \frac{d}{dx} \log(1-x^2) dx \\
&= \frac{\log(1-x^2)}{x} + \int \frac{2}{1-x^2} dx \\
&= \frac{\log(1-x^2)}{x} + \log(1+x) - \log(1-x) + C.
\end{aligned}$$

Hence,

$$\int_0^1 -\frac{\log(1-x^2)}{x^2} dx = \log 2 + \lim_{x \to 1-} \left[\frac{\log(1-x^2)}{x} - \log(1-x) \right] - \lim_{x \to 0+} \frac{\log(1-x^2)}{x} = \log 4.$$

Note that

$$\lim_{x \to 1-} \left[\frac{\log(1-x^2)}{x} - \log(1-x) \right] = \lim_{x \to 1-} \frac{(1-x)\log(1-x) + \log(1+x)}{x} = \log 2 + \lim_{x \to 1-} (1-x)\log(1-x) = \log 2$$

and

$$\lim_{x \to 0+} \frac{\log(1-x^2)}{x} = 0$$

by L'Hospital's rule.
For the second term and third term, we need dilogarithm function $Li_2(z)$ which is defined as

$$Li_2(z) = \int_0^z -\frac{\log(1-x)}{x}dx = \sum_{n=1}^{\infty} \frac{z^n}{n^2}.$$

Using substitution $z = x^2$, the integral of the second term is

$$\int_0^1 \frac{\log(1-x^2)}{x}dx = \int_0^1 \frac{\log(1-z)}{2z}dz = -\frac{1}{2}Li_2(1) = -\frac{\pi^2}{12}.$$

Note that $Li_2(1) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.
For the third term, by using substitution $z = \frac{1+x}{2}$,

$$\begin{aligned} \int_0^1 -\frac{\log(1-x)}{1+x}dx &= \int_{\frac{1}{2}}^1 -\frac{\log(2-2z)}{z}dz \\ &= \int_{\frac{1}{2}}^1 -\frac{\log(1-z)}{z} - \frac{\log 2}{z}dz \\ &= Li_2(1) - Li_2\left(\frac{1}{2}\right) - \log^2 2. \end{aligned}$$

To get the value $Li_2\left(\frac{1}{2}\right)$, we use the identity

$$Li_2(z) + Li_2(1-z) = \frac{\pi^2}{6} - \log z \log(1-z).$$

This identity can be shown easily by differentiating both sides.($Li_2'(z) = -\frac{\log(1-z)}{z}$ by definition) By applying $z = \frac{1}{2}$ in this identity, we get $Li_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{1}{2}\log^2 2$. Therefore the integral of the third term is $Li_2(1) - Li_2\left(\frac{1}{2}\right) - \log^2 2 = \frac{\pi^2}{12} - \frac{1}{2}\log^2 2$.
Finally, the integral of the last term is

$$\int_0^1 -\frac{\log(1+x)}{1+x}dx = \left[-\frac{1}{2}\log^2(1+x)\right]_0^1 = -\frac{1}{2}\log^2 2.$$

Therefore $\frac{S}{2} = \log 4 - \frac{\pi^2}{12} + \left(\frac{\pi^2}{12} - \frac{1}{2}\log^2 2\right) - \frac{1}{2}\log^2 2$ i.e. $S = 4\log 2 - 2\log^2 2$.