2017 SPRING PROBLEM OF THE WEEK
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The answer is negative, and can be actually derived from the following result. log stands for natural logarithm.

Problem. *Let* $\varepsilon \in (0,1)$ *be any constant and* $c := \log \frac{1}{1-\varepsilon}$. *Then for all large* n, $[n]$ *contains a subset* A *of size at most* $3c^{-1}\log n$ *which intersects every arithmetic progression of length at least* εn *in* $[n]$.

SOLUTION. Setting $k_1 = k_1(n) := \lfloor 3c^{-1}\log n \rfloor$ and $k_2 = k_2(n) := \lceil \varepsilon n \rceil$. Obviously $k_1 + k_2 < n$ for all n sufficiently large. For every n, let $\mathscr{F} = \mathscr{F}(n)$ be the family consisting of all arithmetic progressions of length k_2 in $[n]$. Denote

$$s(n) := \#\left\{(A,S) : A \in \binom{[n]}{k_1}, S \in \mathscr{F}, A \cap S = \emptyset\right\}.$$

We have the following claims.

Claim 1. $|\mathscr{F}| \leq n^2$.

Proof. Taking a look at each sequence $S = (a, a+b, \ldots, a+(k_2-1)b) \in \mathscr{F}$, where $a, b \in \mathbb{N}$, we see that $a + (k_2-1)b \leq n$. On the other hand, each pair $(a,b) \in \mathbb{N}^2$ satisfying $a + (k_2-1)b \leq n$ gives exactly one sequence in $\mathscr{F}$, that is

$$(a, a+b, \ldots, a+(k_2-1)b).$$

Therefore, as that $a + (k_2-1)b \leq n$ and that $a, b \in \mathbb{N}$ follow $a, b < n$,

$$|\mathscr{F}| = \#\{(a,b) \in \mathbb{N}^2 : a + (k_2-1)b \leq n\} \leq n^2. \qquad \square$$

Claim 2. *As* n *grows large,*

$$s(n) < \binom{n}{k_1},$$

which implies that there exists an element of $\binom{[n]}{k_1}$ *that intersects every element of* $\mathscr{F}$.

Proof. By counting $s(n)$ with respect to the elements of $\mathscr{F}$ and using Claim 1, we obtain

$$s(n) = |\mathscr{F}|\binom{n-k_2}{k_1} \leq n^2\binom{n-k_2}{k_1}.$$

Now, observe that

$$\frac{\binom{n}{k_1}}{\binom{n-k_2}{k_1}} = \prod_{i=0}^{k_1-1} \frac{n-i}{n-k_2-i} \geq \left(\frac{n}{n-k_2}\right)^{k_1},$$

and

$$\frac{n}{n-k_2} \geq \frac{n}{n-\varepsilon n} = \frac{1}{1-\varepsilon},$$

we have, as $c = \log \frac{1}{1-\varepsilon}$, that for all n large,

$$\frac{\binom{n}{k_1}}{\binom{n-k_2}{k_1}} \geq \frac{1}{(1-\varepsilon)^{k_1}} = e^{ck_1} > n^2,$$

while the last inequality holds because $k_1 = \lfloor 3c^{-1} \log n \rfloor > 2c^{-1} \log n$ for all large n. □

The last claim completes the solution. ❑