KAIST POW 2017-08

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Problem. *Does there exist a constant* $\epsilon > 0$ *such that for each positive integer* n *and each subset* A *of* $\{1, 2, \cdots, n\}$ *with* $|A| < \epsilon n$*, there exists an arithmetic progression* S *in* $\{1, 2, \cdots, n\}$ *such that* $S \cap A \neq \emptyset$ *and* $|S| > \epsilon n$*?*

Solution. We claim that there is no such constant ϵ. To prove this suppose $\epsilon > 0$ is given. Then because there are infinitely many prime numbers, there exists prime p satisfying $\frac{1}{p-1} < \epsilon$. Now consider a subset $A = \{p, 2p, \cdots, p^2\}$ in $\{1, 2, \cdots, p^2\}$. It is clear that $|A| = \frac{1}{p} \cdot p^2 < \epsilon p^2$. Then does there exist an arithmetic progresion S in $\{1, 2, \cdots, n\}$ such that $|S| > \frac{p^2}{p-1}$? For such S we have $|S| > \frac{p^2}{p-1} > p$ so it should contain at least $p+1$ consecutive terms a, $a+d$, $\cdots$, $a+pd$. This is included in $\{1, \cdots, p^2\}$ so we have $pd \leq p^2 - 1$, which means that $d < p$. If so, d is relatively prime with p, and for any $1 < m < p$, dm is also relatively prie with p. Thus, a, $a+d$, $\cdots$, $a+(p-1)d$ contains p different residues modulo p, which includes zero. Thus, one of them are contained in A, a contradiction. Thus there is no such S. ∎