POW-10

Define a function $f : \mathbb{R}^m \times \mathbb{R}^m \cdots \mathbb{R}^m \to \mathbb{R}$ for $n \times m$ matrix M.

$$f_M(x_1, \cdots, x_m) = \prod_{i=1}^{m} |Mx_i|^2$$

Claim. *When $x_1, \cdots, x_m$ forms a orthonormal basis then*

$$f_M(x_1, \cdots, x_m) \geq \det\left(M^T M\right)$$

If $x_1, \cdots, x_m$ are eigenvectors of $M^T M$ then equality holds.

Proof. Let $v_1, \cdots, v_m$ be eigenvectors of $M^T M$ which forms orthonormal basis. Then we can find an orthogonal matrix C such that

$$\left[x_1, \cdots, x_m\right] = \left[v_1, \cdots, v_m\right] C$$

Then

$$|Mx_i|^2 = x_i^T M^T M x_i = \left(c_{1i} v_1^T + \cdots + c_{mi} v_m^T\right)\left(\lambda_1 c_{1i} v_1 + \cdots + \lambda_m c_{mi} v_m\right) = \lambda_1 c_{1i}^2 + \cdots + \lambda_m c_{mi}^2$$

where c_{ij} is i, j th component of C. Now apply weighted AM-GM inequality then we obtain

$$f_M(x_1, \cdots, x_m) = \prod_{i=1}^{m} \left(\lambda_1 c_{1i}^2 + \cdots + \lambda_m c_{mi}^2\right) \geq \prod_{i=1}^{m} \lambda_1^{c_{1i}^2} \cdots \lambda_m^{c_{mi}^2} = \lambda_1 \cdots \lambda_m = \det\left(M^T M\right)$$

Since $\sum_{j=1}^{m} c_{ij}^2 = \sum_{i=1}^{m} c_{ij}^2 = 1$. And trivialy when $x_1, \cdots, x_m$ are eigenvectors of $M^T M$ equality holds. □

Back to problem, denote number of columns of A and B as p, q.

$$M = \left[A, B\right]$$

Denote eigenvectors of A and B, each forming orthonormal basis of $\mathbb{R}^p$ and $\mathbb{R}^q$ as $a_1, \cdots, a_p$ and $b_1, \cdots, b_q$. Now define

$$x_1 = \begin{bmatrix} a_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \cdots, x_p = \begin{bmatrix} a_p \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

with q zeros each. And

$$x_{p+1}=\begin{bmatrix}0\\ \vdots\\ 0\\ b_1\end{bmatrix},\cdots,x_{p+q}=\begin{bmatrix}0\\ \vdots\\ 0\\ b_q\end{bmatrix}$$

with p zeros each. Then $x_1,\cdots,x_{p+q}$ forms an orthonormal basis of $\mathbb{R}^{p+q}$. So by the claim

$$\begin{aligned}\det\left(M^TM\right) &\leq \prod_{i=1}^{p+q}|Mx_i|^2\\ &\leq \prod_{i=1}^{p}|Ma_i|^2\prod_{i=1}^{q}|Mb_i|^2\\ &= \prod_{i=1}^{p}|Aa_i|^2\prod_{i=1}^{q}|Bb_i|^2\\ &= \det\left(A^TA\right)\det\left(B^TB\right)\end{aligned}$$