2017 SPRING PROBLEM OF THE WEEK
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Problem. *Solve the positive Diophantine equation*

$$3^a + 5^b = 2^c. \tag{1}$$

SOLUTION. We have three claims.

Claim 1. *If $b = 1$ then $(a,b,c) = (1,1,3),(3,1,5)$ are solutions.*

Proof. (1) becomes

$$3^a + 5 = 2^c.$$

If $c \le 5$ then we can easily check that $(a,b,c) = (1,1,3),(3,1,5)$ are solutions. If $c \ge 6$, then

$$3^a \equiv -5 \equiv 3^3 \pmod{2^5},$$

following that $8 \mid a-3$ as 3 belongs to 8 modulo 2^5. But 3 belongs to 16 modulo 2^6, and

$$3^a \equiv -5 \not\equiv 3^3 \pmod{2^6},$$

thus $16 \nmid a-3$. Consequently, $a \equiv 11 \pmod{16}$, which implies

$$2^c \equiv 3^a + 5 \equiv 3^6 + 5 \equiv -5 \pmod{17},$$

but it is easy to verify, as 2 belongs to 8 modulo 17, that there is no c such that $2^c \equiv -5 \pmod{17}$.

Therefore, $(a,b,c) = (1,1,3),(3,1,5)$ if $b = 1$. □

Claim 2. *If $a = 1$ then $(a,b,c) = (1,1,3),(1,3,7)$ are solutions.*

Proof. (1) becomes

$$3 + 5^b = 2^c.$$

If $c \le 7$ then $c = 3,7$ satisfy (1), or $(a,b,c) = (1,1,3),(1,3,7)$ are two solutions. If $c \ge 8$, then

$$5^b \equiv -3 \equiv 5^3 \pmod{2^7},$$

implying $32 \mid b-3$, as 5 belongs to 32 modulo 2^7. On the other hand,

$$5^b \equiv -3 \not\equiv 5^3 \pmod{2^8},$$

and 5 belongs to 64 modulo 2^8, thus $64 \nmid b-3$, or $(b-3) \bmod 256 \in \{32, 96, 160, 224\}$, or

$$b \bmod 256 \in \{35, 99, 163, 227\}.$$

Therefore, because 257 is prime, we obtain,

$$2^c - 3 \equiv 5^b \equiv 5^x \pmod{257},$$

where $x \in \{35, 99, 163, 227\}$. Observe that, as 5 belongs to 256 modulo 257 (5 is a quadratic nonresidue modulo 257),

$$5^{163} \equiv -5^{35} \pmod{257}, \quad 5^{227} \equiv -5^{99} \pmod{257}.$$

Furthermore, since $5^7 \equiv -3 \pmod{257}$ and $3^5 \equiv -14 \pmod{257}$, we have

$$5^{35} \equiv (5^7)^5 \equiv (-3)^5 \equiv 14 \pmod{257},$$
$$5^{99} \equiv 5(5^7)^{14} \equiv 5(-3)^{14} \equiv 5 \cdot 3^4 (3^5)^2 \equiv 5 \cdot 81(-14)^2 \equiv -33 \pmod{257},$$

which implies that

$$2^c - 3 \bmod 257 \in \{5^{35}, 5^{99}, 5^{163}, 5^{227}\} = \{14, -33, -14, 33\},$$

or $2^c \bmod 257 \in \{17, -30, -11, 36\}$. However, we can easily check, since 2 belongs to 16 modulo 257, that there is no power of 2 which is congruent to $17, -30, -11$, or 36 modulo 257.

Hence, there is no solution safisfying $c \geq 8$, thus $(a, b, c) = (1, 1, 3), (1, 3, 7)$ if $a = 1$. $\square$

Claim 3. *There is no solution if $a, b \geq 2$.*

Proof. We have $c \geq 3$. Thus, in (1) considering modulo 8 we see that a, b are odd, which follows, by considering modulo 3, that c is also odd. As $a \geq 2$, we have

$$2^c \equiv 5^b \equiv -4^b \equiv -2^{2b} \pmod 9,$$

implying $9 \mid 2^{2b-c} + 1$ or $3 \mid 2b - c$, or $3 \mid b + c$. Thus, $(b, c) \bmod 6 \in \{(3, 3), (5, 1), (1, 5)\}$ as b, c are odd. If $(b, c) \bmod 6$ is either $(3, 3)$ or $(5, 1)$, then $2^c - 5^b \bmod 7$ is either 2 or -1, respectively. The first subcase is impossible since there is no odd a such that $3^a \equiv 2 \pmod 7$. The second subcase leads to $3 \mid a$, but then, as $5^6 \equiv 2^6 \equiv -1 \pmod{13}$ and $3^3 \equiv 1 \pmod{13}$,

$$1 \equiv 3^a \equiv 2^c - 5^b \equiv 2(-1)^{\frac{c-1}{6}} - 5^5(-1)^{\frac{b-5}{6}} \equiv 2(-1)^{\frac{c-1}{6}} - 8(-1)^{\frac{b-5}{6}} \pmod{13},$$

or $1 \bmod 13 \in \{\pm 3, \pm 6\}$, which is absurd. Thus, we obtain $(b, c) \bmod 6 = (1, 5)$. Considering modulo 7 again, we see that

$$3^a \equiv 2^c - 5^b \equiv 2^5 - 5 \equiv -1 \pmod 7,$$

or $a \equiv 3 \pmod 6$. This leads to

$$1 \equiv 3^a \equiv 2^c - 5^b \equiv 2^5(-1)^{\frac{c-5}{6}} - 5(-1)^{\frac{b-1}{6}} \equiv 6(-1)^{\frac{c-5}{6}} - 5(-1)^{\frac{b-1}{6}} \pmod{13},$$

and thus $\frac{c-5}{6}$ and $\frac{b-1}{6}$ are even. We take congruences modulo 5 now to get, as $2^4 \equiv 1 \pmod 5$,

$$3^a \equiv 2^c \equiv 2^5 \equiv 2 \pmod 5,$$

so that $a \equiv 3 \pmod 4$. Consequently, there are nonnegative integers a_1, b_1, c_1 such that

$$a = 12a_1 + 3, \quad b = 12b_1 + 1, \quad c = 12c_1 + 5,$$

and we obtain a new equation

$$3^{12a_1+3} + 5^{12b_1+1} = 2^{12c_1+5}.$$

Since $b \geq 2$,

$$2^{12c_1+5} \equiv 3^{3(4a_1+1)} \equiv 2^{4a_1+1} \pmod{25},$$

so that, as 2 belongs to 20 modulo 25, $5 \mid 4a_1 - 12c_1 - 4$ or $5 \mid a_1 + 2c_1 - 1$, thus

$$(a_1, c_1) \bmod 5 \in \{(2,2),(1,0),(3,4),(4,1),(0,3)\},$$

or, considering modulo 61, by noting that $3^5 \equiv -1 \pmod{61}$ and $2^6 \equiv 3 \pmod{61}$,

$$\begin{aligned} 5^{12b_1+1} \bmod 61 &\in \{2^{12y+5} - 3^{12x+3} : (x,y) \in \{(2,2),(1,0),(3,4),(4,1),(0,3)\}\} \bmod 61 \\ &= \{24, -28, -10, -20, -1\}. \end{aligned}$$

However, this is a contradiction: as 5 belongs to 30 modulo 61 (5 is a quadratic residue modulo 61) and $5^3 \equiv 3 \pmod{61}$ we have

$$\{5^{12x+1} : x \in \mathbb{N}\} \bmod 61 = \{5, 5^{13}, 5^{25}, 5^{37}, 5^{49}\} \bmod 61 = \{5, -22, -13, -16, -15\}.$$

As a consequence, there is no solution if $a, b \geq 2$. □

To sum up, the solutions are $(a,b,c) = (1,1,3),(1,3,7),(3,1,5)$. ❑