2017-07 POW

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- $f(\theta)$ is increasing on $\theta > 0$.

$$f'(\theta) = \sum_{n=1}^{\infty} \left(\frac{3}{(n+3\theta)^2} - \frac{1}{(n+\theta)^2} \right)$$ since this sum uni. conv.

on compact interval since $\displaystyle\sum_{n=N}^{\infty} \frac{1}{(n+t)^2} \le \sum_{n=N}^{\infty} \left(\frac{1}{n+t-1} - \frac{1}{n+t} \right) = \frac{1}{N+t-1} \to 0$

as $N \to \infty$ uniformly on compact interval, and $f(\theta)$ is convergent, $\left(\frac{1}{n+\theta} - \frac{1}{n+3\theta} = \Theta\left(\frac{1}{n^2}\right) \right)$

so We can interchange diff. and infinite sum.

Also, $$\frac{3}{(n+3\theta)^2} \ge \frac{3}{(n+3\theta)} - \frac{3}{(n+3\theta+1)}$$

and $$\frac{1}{(n+\theta)^2} \le \frac{1}{(n+\theta-(1/2))} - \frac{1}{(n+\theta+1/2)}$$

So $$f'(\theta) \ge \frac{3}{1+3\theta} - \frac{1}{1/2+\theta} > 0. \quad \text{for all } \theta > 0.$$

Now $\sup_{\theta>0} f(\theta) = \lim_{M\to\infty} f(M) \quad (M \in \mathbb{N})$.

$$f(M) = \lim_{N\to\infty} \sum_{n=1}^{N} \left(\frac{1}{n+M} - \frac{1}{n+3M} \right) = \lim_{N\to\infty} \sum_{n=M+1}^{3M} \frac{1}{n} - \sum_{n=M+N+1}^{3M+N} \frac{1}{n}$$

$$= \sum_{n=M+1}^{3M} \frac{1}{n}$$ since $\displaystyle\sum_{n=M+N+1}^{3M+N} \frac{1}{n} \le \frac{2M}{M+N+1} \to 0$ as $N \to \infty$.

$$\ln\left(\frac{3M+1}{M+1}\right) = \int_{M+1}^{3M+1} \frac{1}{x}\,dx \le f(M) \le \int_{M}^{3M} \frac{1}{x}\,dx = \ln 3$$

So $$\sup_{\theta>0} f(\theta) = \lim_{M\to\infty} f(M) = \ln 3.$$