POW: 2017-06 Powers of 2

2014[redacted] Jo Tae Hyouk

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All the bases of the log in this proof is 10. For given $r \in \mathbb{R}$, let $\{r\} := r - \lfloor r \rfloor$.
Observe that first digit of 2^n is 9 iff $9 \times 10^m \leq 2^n < 10^{m+1}$ for some nonnegative integer m iff $\log 9 \leq \{n \log 2\} < 1$. Since $\log 2$ is irrational, the set $\{\{n \log 2\} : n \in \mathbb{N}\}$ is dense in $[0, 1]$. Therefore there are infinitely many powers of 2 which starts with 9 in decimal representation.

Lemma. *If $r \in \mathbb{R}$ is irrational, then $S := \{\{nr\} : n \in \mathbb{N}\}$ is dense in $[0, 1]$*

Proof. A set $A \subset [0, 1]$ is dense in $[0, 1]$ iff for all integer $n > 1$, $A \cap [\frac{k}{n}, \frac{k+1}{n}) \neq \emptyset$ for nonnegative integer $k < n$. Let $n > 1$ be an integer. By pigeon-hole principle, there exist two distinct nonnegative integers a and b so that $\frac{k}{n} \leq \{ar\} < \{br\} < \frac{k+1}{n}$ for some nonnegative integer $k < n$. Let $\delta = \{br\} - \{ar\}$. If $a < b$, then $\{m(b-a)r\} = m\delta$ for integer $m \in (0, \delta^{-1})$. If $a > b$, then $\{m(b-a)r\} = 1 - m\delta$ for integer $m \in (0, \delta^{-1})$. Since $0 < \delta < \frac{1}{n}$, for each nonnegative integer $k < n$ there exists two integers $m', m \in (0, \delta^{-1})$ so that $m\delta, 1 - m'\delta$ are in $[\frac{k}{n}, \frac{k+1}{n})$. Therefore S is dense in $[0, 1]$. $\square$