POW: 2017-04 More than a half

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Problem. *Prove or disprove that exactly one of the following is true for every subset A of $\{(i,j) : i,j \in \{1,2,...,n\}, i \neq j\}$.*

(i) There exists a sequence of distinct integers $i_1, i_2, ..., i_k \in \{1,2,...,n\}$ for some integer $k > 1$ such that

$$(i_1, i_2), (i_2, i_3), ..., (i_{k-1}, i_k), (i_k, i_1) \in A.$$

(ii) There exists a collection of finite sets $A_1, ..., A_n$ such that for all distinct $i, j \in \{1,2,...,n\}$,$(i,j) \in A$ if and only if

$$\frac{1}{2}|A_i| < |A_i \cap A_j| \leq \frac{1}{2}A_j.$$

Proof. Suppose A is a subset of $\{(i,j) : i,j \in \{1,2,...,n\}, i \neq j\}$. If A satisfies (i), then (ii) can not be true since if both holds, then

$$|A_{i_1}| < |A_{i_2}| < ... < |A_{i_k}| < |A_{i_1}|$$

which is a contradiction. So suppose (i) does not hold. A can be viewed as directed graph. Since A does not satisfy (i), A is acyclic. Therefore, by using topological sort (click here to see what topological sort is), there is a renumbering of $1, 2, ..., n$ so that $(i,j) \in A$ implies $i < j$. To prove the problem, we give a construction of $A_1, ..., A_n$. Before giving the construction, we start with some notions. For two set X and Y, let $X + Y := X \cup Y$ and $X - Y := X \setminus Y$. For more than two sets $X_1, ..., X_n$, let

$$X_1 \pm X_2 ... \pm X_n := (...((X_1 \pm X_2) \pm X_3)...) \pm X_n)$$

so the calculation goes from left to right. And let I_j^k be 1 if $(j,k) \in A$ and -1 else. With these notions, now we give the construction of $A_1, ..., A_n$ as follows: Let $A_1 = \{1\}$. For $1 < k \leq n$, define A_k to be the set

$$A_k = \{1, 2, 3, ..., 3^{k-1}\} + I_{k-1}^k A_{k-1} + I_{k-2}^k A_{k-2} + ... + I_1^k A_1$$

Observe that (recall $(j,k) \in A$ implies $j < k$)

1. $3^{k-2} + 1, ..., 3^{k-1}$ are in A_k, so $3^{k-1} - 3^{k-2} \leq |A_k| \leq 3^{k-1}$
2. If $(j,k) \in A$, then $3^{j-2} + 1, ..., 3^{j-1}$ are also in $A_j \cap A_k$
3. If $(j,k) \notin A$, then $3^{j-2} + 1, ..., 3^{j-1}$ are also not in $A_j \cap A_k$

By these observations, if $(j,k) \in A$,

$$\frac{|A_j|}{2} \leq \frac{3^{j-1}}{2} < 3^{j-1} - 3^{j-2} \leq |A_j \cap A_k| \leq |A_j| \leq \frac{|A_k|}{2}$$

and if $(j,k) \notin A$,

$$|A_j \cap A_k| \leq 3^{j-2} \leq \frac{|A_j|}{2}$$

Therefore (ii) holds. □