2017-06 Powers of 2

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pf)

Suppose there exists finite n such that first digit of 2^n is 9 and let the maximum such n as M. (There exists such n like $n = 53$ and $2^{53} = 9007199254740992$.)

Suppose, $S = \{\ \{log\ 2^x\}\ |x > M\}$ and $U = lub\ S$. Here $\{\}$ denotes fractional part ($\{x\} = x - \lfloor x \rfloor$). Then $U \leq log\ 9$, since 10^{u+n} where $log\ 9 \leq u < 1, n \in \mathbb{N}$ will give 9 as first digit.

Suppose, $T = \{log\ 1.024 + \{log\ 2^x\}|x > M\}$. Then, $lub\ T = lub\ U + log\ 1.024 < log\ 9 + log\ 1.024 = log\ 9.216 < 1$. Therefore, we also can write T as $T = \{\ \{log\ 2^{10}2^x\}\ |x > M\} = \{\ \{log\ 2^x\}\ |x > M + 10\} \subset S$.

$T \subset S$ but $lub\ T > lub\ U$ gives contradiction. Therefore, there exists infinitely many n such that first digit of 2^n is 9. $\square$

Note)

Also, we can explicitly give sequence $a_1 = 2^{53}$, $a_{n+1} = a_n * 2^{93}$ when second digit of a_n is larger than or equal to 5, $a_{n+1} = a_n * 2^{10}$, when second digit of a_n is less than or equal to 4.

$9.5 \times 10^k \leq a_n < 10.0 \times 10^k$ gives $9.4 \times 10^{k+28} \leq a_{n+1} < 10.0 \times 10^{k+28}$. $(0.99 \times 10^{28} < 2^{93} < 10^{28})$

$9.0 \times 10^k \leq a_n < 9.5 \times 10^k$ gives $9.0 \times 10^{k+3} \leq a_{n+1} < 9.8 \times 10^{k+3}$. $(10^3 < 2^{10} < 1.03 \times 10^3)$

Using induction gives first digit of every element in $\{a_n\}$ is 9, (2^{53}, 2^{63}, 2^{73}, 2^{83}, 2^{176}, 2^{269}, 2^{279}, $\cdots$).