2017 SPRING PROBLEM OF THE WEEK
POW2017-05

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Problem. *Suppose that* $f:(2,\infty)\to(-2,2)$ *is a continuous function and there exists a positive constant* m *such that* $|1+xf(x)+(f(x))^2|\le m$ *for any* $x>2$*. Prove that, for any* $x>2$,

$$\left|f(x)-\frac{-x+\sqrt{x^2-4}}{2}\right|\le 6\sqrt{m}.$$

By substituting $t:=\frac{x-\sqrt{x^2-4}}{2}\in(0,1)$ and denote $g(t):=-f(t+\frac{1}{t})=-f(x)$, we will prove a similar result, but much stronger.

Problem. *Let* $g:(0,1)\to(-2,2)$ *be a continuous function such that there exists a positive real* m *satisfying*

$$\left|(g(t)-t)\left(g(t)-\frac{1}{t}\right)\right|\le m\quad\forall t\in(0,1).$$

Prove that

$$|g(t)-t|\le(1+\sqrt{2})\sqrt{m}\quad\forall t\in(0,1).$$

SOLUTION. Setting $c:=1+\sqrt{2}$, then $c-\frac{1}{c}=2$. Suppose, for the sake of contradiction, that there is some $a\in(0,1)$ such that $|g(a)-a|>c\sqrt{m}$. We have some claims below.

Claim 1. $2g(a)>a+\frac{1}{a}$.

Proof. From the contradictive hypothesis, we have

$$|g(a)-a|>c\sqrt{m}>\frac{m}{c\sqrt{m}}>\frac{|g(a)-a|\left|g(a)-\frac{1}{a}\right|}{|g(a)-a|}=\left|g(a)-\frac{1}{a}\right|.$$

We find out $(g(a)-a)^2>(g(a)-\frac{1}{a})^2$, or

$$\left(\frac{1}{a}-a\right)\left(2g(a)-a-\frac{1}{a}\right)>0.$$

As $\frac{1}{a}>1>a$, we obtain $2g(a)>a+\frac{1}{a}$. □

Claim 2. $a<\sqrt{m+1}-\sqrt{m}$.

Proof. From Claim 1, $g(a) > \frac{1}{2}(a + \frac{1}{a}) > a$, which follows $g(a) - a = |g(a) - a| > c\sqrt{m}$. On the other hand,

$$g(a) - \frac{1}{a} \le \left| g(a) - \frac{1}{a} \right| \le \frac{m}{|g(a) - a|} < \frac{\sqrt{m}}{c},$$

therefore

$$\frac{1}{a} - a > \left(g(a) - \frac{\sqrt{m}}{c} \right) - (g(a) - c\sqrt{m}) = \left(c - \frac{1}{c} \right) \sqrt{m} = 2\sqrt{m},$$

and then we easily have $a < \sqrt{m+1} - \sqrt{m}$. $\square$

Claim 3. There is some $t_0 \in (0, \sqrt{m+1} - \sqrt{m})$ such that $2g(t_0) = t_0 + \frac{1}{t_0}$.

Proof. For any $t \in (0, \frac{1}{2})$, as $g(t) \in (-2, 2)$, $\frac{1}{t} > 2 > g(t)$. Thus

$$|g(t) - t| \le \frac{m}{\frac{1}{t} - g(t)} < \frac{m}{\frac{1}{t} - 2} \quad \forall t \in \left(0, \frac{1}{2} \right).$$

This implies $\lim_{t \to 0^+} |g(t) - t| = 0$ or $\lim_{t \to 0^+} g(t) = 0$. Moreover, since $\lim_{t \to 0^+} \left(t + \frac{1}{t} \right) = \infty$, there exists $b \in (0, a)$ small enough such that $2g(b) < b + \frac{1}{b}$. Considering the function $h(t) := 2g(t) - t - \frac{1}{t}$, we see that h is continuous on $(0, 1)$ while $h(b) < 0$ and $h(a) > 0$. Thus, there is some $t_0 \in (b, a)$ such that $h(t_0) = 0$, or $2g(t_0) = t_0 + \frac{1}{t_0}$. Because $a < \sqrt{m+1} - \sqrt{m}$ by Claim 2, $t_0 \in (0, \sqrt{m+1} - \sqrt{m})$. $\square$

Claim 4. $|g(t_0) - t_0||g(t_0) - \frac{1}{t_0}| > m$, which gives the contradiction.

Proof. From Claim 3, we have

$$|g(t_0) - t_0| = \frac{1}{2} \left(\frac{1}{t_0} - t_0 \right) = \left| g(t_0) - \frac{1}{t_0} \right|,$$

and

$$\frac{1}{t_0} - t_0 > \frac{1}{\sqrt{m+1} - \sqrt{m}} - (\sqrt{m+1} - \sqrt{m}) = 2\sqrt{m}.$$

Thus

$$|g(t_0) - t_0| \left| g(t_0) - \frac{1}{t_0} \right| = \frac{1}{4} \left(t_0 - \frac{1}{t_0} \right)^2 > \frac{1}{4} (2\sqrt{m})^2 = m. \quad \square$$

Therefore, there is no $a \in (0, 1)$ such that $|g(a) - a| > c\sqrt{m}$. The solution is completed. ❑