KAIST POW 2017-03

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Problem. *For an integer $n \geq 4$, find the solutions of the equation*

$$\sum_{k=1}^{n} \frac{\sin \frac{k\pi}{n+1}}{\sin\left(\frac{k\pi}{n+1} - x\right)} = 0.$$

Solution. We observe that

$$\sum_{k=1}^{n} \frac{\sin \frac{k\pi}{n+1}}{\sin\left(\frac{k\pi}{n+1} - x\right)} = \sum_{k=1}^{n} \frac{e^{2k\pi i/(n+1)} - 1}{e^{2k\pi i/(n+1)-2ix} - 1} \cdot e^{-ix} = 0 \Leftrightarrow \sum_{k=1}^{n} \frac{e^{2k\pi i/(n+1)} - 1}{e^{2k\pi i/(n+1)-2ix} - 1} = 0.$$

Here, for each $k = 1, \cdots, n$, we define

$$w_k = \frac{e^{2k\pi i/(n+1)} - 1}{e^{2k\pi i/(n+1)-2ix} - 1}, \text{ i.e., } e^{2k\pi i/(n+1)} = \frac{w_k - 1}{e^{-2ix}w_k - 1}.$$

Because $e^{2k\pi i/(n+1)}$ are distinct, w_k are also disticnt. Moreover, w_k are nonzero because $e^{2k\pi i/(n+1)} - 1 \neq 0$. Note that $0, w_1, \cdots, w_n$ are solutions of $(x-1)^{n+1} = (e^{-2ix}x-1)^{n+1}$.

Unless $(e^{-2ix})^{n+1} = 1$, $(x-1)^{n+1} - (e^{-2ix}x - 1)^{n+1}$ becomes a polynomial of degree $n+1$ with zeroes 0, w_1, $\cdots$, w_n. Sum of these numbers are $n - e^{-2inx}n$, which becomes 0 when $x = \frac{t\pi}{n}$ for some $t \in \mathbb{Z} \setminus n\mathbb{Z}$. When $(e^{-2ix})^{n+1} = 1$, i.e. $x = m\pi/(n+1)$ for some $m \in \mathbb{Z}$, original sum is either not defined, becomes m or $-m$, so it cannot be a solution. Thus, the solutions are $x = \frac{t\pi}{n}$ for $t \in \mathbb{Z} \setminus n\mathbb{Z}$.

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