KAIST POW 2017-02

15학번 유찬진

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Problem. Let a_1, a_2, $\cdots$, a_n be distinct points in $\mathbb{R}^4$. Does there exist a non-zero polynomial $p(x_1, x_2, x_3, x_4)$ such that $\deg p \le \lceil \sqrt{5} n^{1/4} \rceil$ and $p(a_i) = 0$ for all $i = 1, 2, \cdots, n$?

Solution. Let $P_m(x_1, x_2, x_3, x_4)$ be the set of all real polynomials in variables x_1, x_2, x_3, and x_4 whose degree is less than or equal to m. It is a vector space over $\mathbb{R}$ with dimension

$$\left(\!\!\binom{5}{m}\!\!\right) = \binom{5+m-1}{m} = \frac{1}{24}(m+4)(m+3)(m+2)(m+1),$$

since

$$\left\{ \prod_{k=1}^{4} x_k^{i_k} \ : \text{for each } k,\ i_k \ge 0 \text{ and } \sum_{j=1}^{4} i_j \le m \right\}$$

is a basis for $P_n(x_1, x_2, x_3, x_4)$ over $\mathbb{R}$. Note that each element corresponds to a nonnegative integral solution $(i_1, i_2, i_3, i_4, i_5)$ of $\sum_{j=1}^{5} i_j = m$.

Let $m = \lceil \sqrt{5} n^{1/4} \rceil$. The dimension of $P_m(x_1, x_2, x_3, x_4)$ is

$$d := \frac{1}{24}(m+4)(m+3)(m+2)(m+1) > \frac{m^4}{25} \ge n.$$

Denote the basis elements by p_1, $\cdots$, p_d. Any $p \in P_m(x_1, x_2, x_3, x_4)$ can be written as $p = \sum_{j=1}^{d} c_j p_j$ where $c_j \in \mathbb{R}$. If it is applied to a_i, $i = 1, \cdots, n$, we obtain a system of linear equations $A\mathbf{x} = \mathbf{0}$, where $A = [p_j(a_i)]_{n \times d}$ and $\mathbf{x} = [c_j]_{d \times 1}$. Since $d > n$, $\operatorname{rank} A \le \min\{d, n\} < d$. It follows that the system has a nontrivial solution $\mathbf{x} = [c_j]_{d \times 1}$. The polynomial p determined by these c_j, $j = 1, \cdots, d$, satisfies all the conditions, so we have a positive answer to the question. □