2017 Spring POW Week #1 (2017-01)

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Problem 1. *Let A, B, C be $N \times N$ Hermitian matrices with $C = A + B$. Let $\alpha_1 \geq \cdots \geq \alpha_N, \beta_1 \geq \cdots \geq \beta_N, \gamma_1 \geq \cdots \geq \gamma_N$ be the eigenvalues of A, B, C, respectively. For any $1 \leq k \leq N$, prove that*

$$\gamma_1 + \gamma_2 + \cdots + \gamma_k \leq (\alpha_1 + \alpha_2 + \cdots \alpha_k) + (\beta_1 + \beta_2 + \cdots + \beta_k).$$

Solution. Let Γ_k^N be the set of all k-tuples of orthonormal vectors in $\mathbb{C}^N$, i.e., $\Gamma_k^N = \{(v_1, \ldots, v_k) \in (\mathbb{C}^N)^k : v_i^* v_j = \delta_{ij} \text{ for all } i, j\}$, where δ_{ij} is kronecker delta. Now we claim that if P is $N \times N$ Hermitian matrices then $\max_{(v_1,\ldots,v_k)\in\Gamma_k^N} \sum_{i=1}^k v_i^* P v_i$ exists and is equal to the sum of k largest eigenvalues of P. If this claim is true, then we can easily prove the goal inequality by

$$\max_{(v_1,\ldots,v_k)\in\Gamma_k^N} \sum_{i=1}^k v_i^* C v_i \leq \max_{(v_1,\ldots,v_k)\in\Gamma_k^N} \sum_{i=1}^k v_i^* A v_i + \max_{(v_1,\ldots,v_k)\in\Gamma_k^N} \sum_{i=1}^k v_i^* B v_i$$
$$\Rightarrow \gamma_1 + \gamma_2 + \cdots + \gamma_k \leq (\alpha_1 + \alpha_2 + \cdots + \alpha_k) + (\beta_1 + \beta_2 + \cdots + \beta_k).$$

At first, we need to mention that for any $v \in \mathbb{C}^N$, we have $\overline{v^* P v} = (v^* P v)^* = (v^* P^* v) = (v^* P v)$ so $(v^* P v) \in \mathbb{R}$. Additionally, the set Γ_k^N is a compact subset of $(\mathbb{C}^N)^k$ and the mapping $(v_1, \ldots, v_k) \mapsto \sum_{i=1}^k v_i^* P v_i$ is continuous. It implies the value $\max_{(v_1,\ldots,v_k)\in\Gamma_k^N} \sum_{i=1}^k v_i^* P v_i$ exists. Now we may assume that P is real diagonal matrix (the diagonal entries are exactly the list of the eigenvalues of P), because if Q is $N \times N$ unitary matrix and $(v_1, \ldots, v_k) \in \Gamma_k^N$ then $(Qv_1, \ldots, Qv_k) \in \Gamma_k^N$ holds. Moreover, we may assume the diagonal entries are sorted in descending order.

Let $v_i = (v_{i1}, \ldots, v_{iN})$ for each $i = 1, \ldots, k$ then we have $1 = \|v_i\|^2 = \sum_{j=1}^N |v_{ij}|^2$ for each $i = 1, \ldots, k$. Also one can observe that $\sum_{i=1}^k |v_{ij}|^2 \leq 1$ for each $j = 1, \ldots, N$. It is because if we make a $N \times N$ unitary matrix U which has $v_1, \ldots, v_k$ as columns then $U^*U = I = UU^*$ so every columns of U^* are orthonormal. Let $\lambda_1 \geq \cdots \geq \lambda_N$ be the diagonal entries of P and let $l_j = \sum_{i=1}^k |v_{ij}|^2$ for $j = 1, \ldots, N$ then $\sum_{i=1}^k v_i^* P v_i = \sum_{j=1}^N \lambda_j l_j, \sum_{j=1}^N l_j = k, 0 \leq l_j \leq 1$ for all $j = 1, \ldots, N$. By greedy algorithm, the value $\sum_{i=1}^k v_i^* P v_i = \sum_{j=1}^N \lambda_j l_j$ is maximized when $l_1 = \ldots, l_k = 1, l_{k+1} = 0, \ldots, l_N = 0$. Actually, if we set $v_i = e_i$ for $i = 1, \ldots, k$ then the value $\sum_{i=1}^k v_i^* P v_i$ is maximized and equal to $\sum_{j=1}^k \lambda_j$. It proves the claim is true, and ends the proof.

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