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Problem. Suppose that $z_1, \ldots, z_n$ are complex numbers satisfying $\sum_{k=1}^{n} z_k = 0$. Prove that

$$\sum_{k=1}^{n} |z_{k+1} - z_k|^2 \geq 4\sin^2\left(\frac{\pi}{n}\right) \sum_{k=1}^{n} |z_k|^2.$$

where we let $z_{n+1} = z_1$.

proof) We may let $n \geq 3$ because the inequality is trivival when $n = 1, 2$. We will use the Fourier transform on a finite abelian group. Set $G = \mathbb{Z}/n\mathbb{Z} = \widehat{G}$ where $\widehat{G}$ is the Pontryagin dual of G. Define $f, g \in L^2(G)$ by $f(k) := z_k$, $g(k) := z_{k+1} - z_k$ and observe that

$$\widehat{g}(\xi) = (e^{\frac{2\pi i \xi}{n}} - 1)\widehat{f}(\xi) \qquad \forall \xi \in \widehat{G}.$$

Also, since we are given that $\widehat{f}(0) = \dfrac{1}{n}\sum_{k=1}^{n} z_k = 0$, we have

$$\sum_{\xi \in \widehat{G}} |\widehat{g}(\xi)|^2 \geq \inf_{\xi \in \widehat{G} \setminus \{0\}} |e^{\frac{2\pi i \xi}{n}} - 1|^2 \cdot \sum_{\xi \in \widehat{G} \setminus \{0\}} |\widehat{f}(\xi)|^2 = 4\sin^2\left(\frac{\pi}{n}\right) \sum_{\xi \in \widehat{G}} |\widehat{f}(\xi)|^2.$$

Applying the Plancherel theorem completes the proof.

Remark. By inspecting the above inequality, equality holds if and only if "$\widehat{f}(\xi) = 0$ for all $\xi \neq 1, n-1$". Since the Fourier transform $\mathscr{F} : L^2(G) \to L^2(\widehat{G})$ is invertible, the set of all such functions forms a two dimensional subspace of $L^2(G)$. In fact, it has a basis $\{f_1, f_2\}$ where

$$f_1 : k \mapsto e^{\frac{2\pi i k}{n}}, \quad \text{and} \quad f_2 : k \mapsto e^{-\frac{2\pi i k}{n}} \qquad \forall k \in G.$$

Therefore, the equality holds if and only if

$$z_k = ce^{\frac{2\pi i k}{n}} + de^{-\frac{2\pi i k}{n}} \qquad \forall k = 1, \ldots, n$$

for some $c, d \in \mathbb{C}$.