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2015 신준형

The problem can be interpreted into a language of linear algebra:

When there is a $(2n+1)\times(2n+1)$ matrix A s.t. all diagonal entries are zero, and for each row there are exactly n 1's and n (-1)'s, the only eigenvector corresponds to 0 is $t(\overbrace{1,\cdots,1}^{2n+1})$, $t\in\mathbb{Z}$.

Now, view A as a linear operator on $\mathbb{Q}^n$, a vector space over the field $\mathbb{Q}$. We will show that $\text{rank}(A)=2n$, so A has one dimensional null space, which completes the proof.

Let r_i be the ith row vector of A, i.e.

$$r_i = (\varepsilon_{i1}, \varepsilon_{i2}, \cdots, \varepsilon_{i(i-1)}, 0, \varepsilon_{i(i+1)}, \cdots, \varepsilon_{in})$$

where $\varepsilon_{ij}\in\{-1,1\}$. We claim that $r_1,\cdots,r_{2n}$ are linearly independent over $\mathbb{Q}$. Assume $c_1r_1+\cdots+c_{2n}r_{2n}=0$, where some of c_i's are nonzero. Multiplying a common multiple of the denominators of c_i's, it can be assume that they are integers and nonzero coefficients are relatively prime.

Then we get the following system viewed modulo 2:

$$\sum_{i=1}^{2n} c_i \equiv 0, \quad \sum_{i\neq j} c_i \equiv 0 \quad (1\le j\le 2n)$$

Hence we must have all coefficients even, which contradicts to our assumption. Hence $r_1,\cdots,r_{2n}$ are linearly independent over $\mathbb{Q}$, so $\text{rank}(A)=2n$.

So the null space of A has dimension 1, hence is generated by $(1,1,\cdots,1)$.

This completes the proof.