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If $n = 0$ there is nothing to prove, so assume $n \geq 1$. Before proving the statement, we will observe the following lemma.

Lemma 1. *Let $V_1, V_2, \cdots$ be countably many k-dimensional subspaces of $\mathbb{R}^n$ for $k < n$. Then there is a vector $\mathbf{x}$ such that $\mathbf{x} \notin V_i$ for $i \geq 1$.*

Proof. We proceed by induction on n. If $n = 1$, then simply take $\mathbf{x}$ as any nonzero vector. Now assume inductively for $n - 1$. Extend each subspace $V_1, V_2, \cdots$ with some vectors so that $\dim V_i = n - 1$ for $i \geq 1$. Then V_i can be represented in the form of $\mathbf{a_i} \cdot \mathbf{x} = 0$, where $||\mathbf{a_i}|| = 1$. Note that the collection is countable where the set of vectors with $||\mathbf{a}|| = 1$ is uncountable. Hence, we can take $\mathbf{a_0}$ so that it does not appear in the collection of $\mathbf{a_i}$'s and $||\mathbf{a_0}|| = 1$. Then the $(n-1)$-dimensional subspace V_0 represented by $\mathbf{a_0} \cdot \mathbf{x} = 0$ differs from any of the subspaces given. Hence, a subspace $W_i = V_0 \cap V_i$ has dimension $n-2$ and is contained in V_0. Note that V_0 becomes identical to $\mathbb{R}^{n-1}$ by applying some rigid motions. Hence by the inductive hypothesis, there is a vector $\mathbf{x} \in V_0$ such that $\mathbf{x} \notin W_i$ for $i \geq 1$. Then $\mathbf{x} \notin V_i$ for each i, completing the proof of the lemma. □

Now we prove the original statement. We will proceed by induction on k, but in reverse order. If $k = n$ then just take the trivial subspace $W = \{0\}$. Now assume inductively for $k + 1$. Let $\mathbf{x}$ be a vector produced by Lemma 1. Then the dimension of the subspace W_i spanned by V_i and $\mathbf{x}$ is $k + 1$. Thus by the inductive hypothesis, there is an $(n - 1 - k)$-dimensional subspace W_0 with $\dim W_i \cap W_0 = 0$ for $i \geq 1$. Hence, the subspace W spanned by W_0 and $\mathbf{x}$ is $(n - k)$-dimensional and $\dim W \cap V_i = 0$ as desired.