POW 2016-19 Zeta function

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Let

$$P(k)=\sum_{i_1=1}^{\infty}\sum_{i_2=1}^{\infty}\cdots\sum_{i_k=1}^{\infty}\frac{1}{i_1 i_2\cdots i_k(i_1+i_2+\cdots+i_k)}$$

for a positive integer k. Find $\zeta(k+1)/P(k)$, where ζ is the Riemann-zeta function.

sol)

We need some lemmas

Lemma 1.

$$\frac{1}{i}=\int_0^1 x^{i-1}dx \qquad (i\in\mathbb{N})$$

Lemma 2.

$$\int_0^1\frac{1}{1-xy}dx=-\frac{1}{y}\log(1-y)$$

Lemma 3.

$$\int_0^\infty\frac{t^k}{e^t-1}dt=\Gamma(k+1)\zeta(k+1) \qquad (k\in\mathbb{N})$$

where $\Gamma(z)=\int_0^\infty e^{-t}t^{z-1}dt$, the Gamma function

pf) $\int_0^\infty e^{-nt}t^{z-1}dt=\frac{1}{n^z}\int_0^\infty e^{-t}t^{z-1}dt=\frac{\Gamma(z)}{n^z}$, so that

$$\Gamma(z)\sum_{n=1}^{\infty}\frac{1}{n^z}=\int_0^\infty t^{z-1}\sum_{n=1}^{\infty}e^{-nt}dt=\int_0^\infty\frac{t^{z-1}}{e^t-1}dt$$

$$\therefore\int_0^\infty\frac{t^k}{e^t-1}dt=\Gamma(k+1)\zeta(k+1)$$

$$
\begin{aligned}
P(k) &= \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{\infty}\cdots\sum_{i_k=1}^{\infty}\frac{1}{i_1 i_2\cdots i_k(i_1+i_2+\cdots+i_k)}\\
&= \sum_{i_1}\cdots\sum_{i_k}\int_0^1\cdots\int_0^1 x_1^{i_1-1}\cdots x_k^{i_k-1}y^{i_1+\cdots+i_k-1}\ dx_1\cdots dx_k dy\\
&= \int_0^1\cdots\int_0^1\sum_{i_1}\cdots\sum_{i_k}(x_1y)^{i_1-1}\cdots(x_ky)^{i_k-1}\ y^{k-1}\ dx_1\cdots dx_k dy\\
&= \int_0^1\cdots\int_0^1\sum_{i_1}\cdots\sum_{i_k}(x_1y)^{i_1-1}\cdots(x_ky)^{i_k-1}\ y^{k-1}\ dx_1\cdots dx_k dy\\
&= \int_0^1\cdots\int_0^1\frac{1}{1-x_1y}\cdots\frac{1}{1-x_ky}\ y^{k-1}\ dx_1\cdots dx_k dy\\
&= \int_0^1\prod_{i=1}^{k}\int_0^1\frac{1}{1-x_iy}dx_i\ y^{k-1}\ dy\\
&= \int_0^1\frac{(-1)^k\,(\log(1-y))^k}{y}\ dy\\
&= \int_0^{\infty}\frac{t^k}{1-e^{-t}}e^{-t}dt\\
&= \int_0^{\infty}\frac{t^k}{e^t-1}dt\\
&= \Gamma(k+1)\zeta(k+1)\\
&= k!\zeta(k+1)
\end{aligned}
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\therefore\ \zeta(k+1)/P(k) = \frac{1}{k!}
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