POW 2016-17 Integral with two variables 2016 최대범

Let $S = 2\pi^2 \log\left[\dfrac{(z+R(z))(w+R(w))}{2(zw-4+R(z)R(w))}\right]$.

$$\frac{\partial^2 L}{\partial z \partial w} = \int_{-2}^{2}\int_{-2}^{2}\left(\frac{1}{z-x}-\frac{1}{z-y}\right)\left(\frac{1}{w-x}-\frac{1}{w-y}\right)\frac{4-xy}{(x-y)^2\sqrt{4-x^2}\sqrt{4-y^2}}\,dx\,dy$$

$$= \int_{-2}^{2}\int_{-2}^{2}\frac{1}{(z-x)(w-x)\sqrt{4-x^2}}\cdot\frac{1}{(z-y)(w-y)\sqrt{4-y^2}}\cdot(4-xy)\,dx\,dy$$

$$= \left(\int_{-2}^{2}\frac{1}{(z-x)(w-x)\sqrt{4-x^2}}dx\right)^2\times 4 - \left(\int_{-2}^{2}\frac{x}{(z-x)(w-x)\sqrt{4-x^2}}dx\right)^2.$$

$$\int_{-2}^{2}\frac{x^n}{(z-x)(w-x)\sqrt{4-x^2}}dx = \frac{1}{w-z}\left(\int_{-2}^{2}\frac{x^n}{(z-x)\sqrt{4-x^2}}dx - \int_{-2}^{2}\frac{x^n}{(w-x)\sqrt{4-x^2}}dx\right) \quad (n\in N^+)$$

$$\int_{-2}^{2}\frac{x^n}{(z-x)\sqrt{4-x^2}}dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{(2\sin t)^n}{z-2\sin t}dt = \frac{1}{2}\int_{|s|=1}\frac{\left(\frac{s-\frac{1}{s}}{i}\right)^n}{z-\frac{s-\frac{1}{s}}{i}}\frac{ds}{is}$$

$$= \frac{1}{2}\int_{|s|=1} - \frac{\left(\frac{s+\frac{1}{s}}{i}\right)^n}{s^2-(iz)s-1}ds = \frac{1}{2}\int_{|s|=1} - \frac{\left(\frac{s+\frac{1}{s}}{i}\right)^n}{\left(s-\frac{iz+\sqrt{4-z^2}}{2}\right)\left(s-\frac{iz-\sqrt{4-z^2}}{2}\right)}ds.$$

For $z=ki$ $(k>0)$, $\left|\dfrac{iz+\sqrt{4-z^2}}{2}\right| = \dfrac{1}{2}\left(-k+\sqrt{4+k^2}\right) = \dfrac{1}{2}\cdot\left(\dfrac{4}{k+\sqrt{4+k^2}}\right) < 1$.

since $\left|\dfrac{iz+\sqrt{4-z^2}}{2}\cdot\dfrac{iz-\sqrt{4-z^2}}{2}\right| = 1$, $\left|\dfrac{iz-\sqrt{4-z^2}}{2}\right| > 1$.

So, $$\int_{-2}^{2}\frac{x^n}{(z-x)\sqrt{4-x^2}}dx = \pi i \operatorname*{Res}_{s=\frac{iz+\sqrt{4-z^2}}{2}} - \frac{\left(\frac{s+\frac{1}{s}}{i}\right)^n}{s^2-(iz)s-1} = -\pi i\frac{z^n}{\sqrt{4-z^2}}$$

$$= -\pi i\frac{z^n}{\sqrt{k^2+4}} = \pi\frac{z^n}{\sqrt{-k^2-4}} = \pi\frac{z^n}{R(z)}.$$

Since $\dfrac{x^n}{(z-x)\sqrt{4-x^2}}$ is analytic for each $x\in[-2,2]$ and $z\in\mathbb{C}-(-\infty,2]$,

So $\displaystyle\int_{-2}^{2}\frac{x^n dx}{(z-x)\sqrt{4-x^2}}$ is analytic on $z\in\mathbb{C}-(-\infty,2]$ and $\pi\dfrac{z^n}{R(z)}$, too, so

$\displaystyle\int_{-2}^{2}\frac{x^n dx}{(z-x)\sqrt{4-x^2}} = \pi\frac{z^n}{R(z)}$ by analytic continuation since they coincide when $z=ki$ $(k>0)$.

So, $$\frac{\partial^2 L}{\partial z\partial w} = \frac{\pi^2}{(w-z)^2}\left(4\cdot\left(\frac{1}{R(z)}-\frac{1}{R(w)}\right)^2 - \left(\frac{z}{R(z)}-\frac{w}{R(w)}\right)^2\right)$$

$$= \frac{\pi^2}{(w-z)^2}\,\frac{4\cdot R(z)^2+4\cdot R(w)^2 - z^2R(w)^2 - w^2R(z)^2 - 8R(z)R(w) + 2zwR(z)R(w)}{R(z)^2R(w)2}$$

$$= \frac{\pi^2}{(w-z)^2}\,\frac{-2R(z)^2R(w)^2 + 2R(z)R(w)(zw-4)}{R(z)^2R(w)^2} = -\frac{2\pi^2}{(w-z)^2}\cdot\frac{R(z)R(w)-(zw-4)}{R(z)R(w)}.$$

$$\frac{\partial S}{\partial w} = 2\pi^2\left(\frac{1+R'(w)}{w+R(w)} - \frac{z+R(z)R'(w)}{zw-4+R(z)R(w)}\right) = 2\pi^2\left(\frac{1}{R(w)} - \frac{1}{R(w)}\frac{R(w)z+wR(z)}{zw-4+R(z)R(w)}\right)$$

$$= \frac{2\pi^2}{R(w)}\left(1-\frac{(R(w)z+wR(z))(zw-4-R(z)R(w))}{(zw-4)^2-(R(z)R(w))^2}\right) = \frac{2\pi^2}{R(w)}\left(1-\frac{wR(w)(z^2-R(z)^2)+zR(z)(w^2-R(w)^2)-4(R(w)z+R(z)w}{4(z-w)^2}\right.$$

$$= \frac{2\pi^2}{R(w)}\left(1-\frac{(R(w)-R(z))(w-z)}{(z-w)^2}\right) = \frac{2\pi^2}{R(w)}\left(1-\frac{R(z)-R(w)}{z-w}\right)$$

$$\frac{\partial^2 S}{\partial z \partial w} = \frac{2\pi^2}{R(w)}\left(\frac{R(z)-R(w)}{(z-w)^2} - \frac{1}{z-w}\cdot\frac{z}{R(z)}\right) = \frac{2\pi^2}{R(w)R(z)}\left(\frac{R(z)^2 - R(z)R(w) - z^2 + zw}{(z-w)^2}\right)$$

$$= -\frac{2\pi^2}{(w-z)^2}\frac{R(z)R(w) - (zw-4)}{R(z)R(w)} = \frac{\partial^2 L}{\partial z \partial w}$$

$\Rightarrow \dfrac{\partial S}{\partial w} = \dfrac{\partial L}{\partial w} + f(w)$ for some Analytic f. Now fix $w \in \mathbb{C} - (-\infty, 2]$.

$$\left|\frac{\partial L}{\partial w}\right| = \left|\int_{-2}^{2}\int_{-2}^{2} \frac{\log(z-x) - \log(z-y)}{x-y}\cdot\left(\frac{1}{w-x} - \frac{1}{w-y}\right)\cdot\frac{4-xy}{\sqrt{4-x^2}\sqrt{4-y^2}}\,dx\,dy\right|.$$

Since w is fixed and $w \in \mathbb{C} - (-\infty, 2]$, There exist M s.t. $\left|\frac{1}{w-x} - \frac{1}{w-y}\right| \le M$ for all $x,y \in [-2,2]\times[-2,2]$. By MVT, $\frac{\log(z-x) - \log(z-y)}{x-y} = \frac{1}{z - c(x,y,z)}$ for some $c(x,y,z) \in (-2,2)$. (This is true even if $x=y$!).

Consider $\lim_{z\to\infty} \frac{\partial L}{\partial w}$, when $z \in \mathbb{R} - (-\infty, 2]$. Then

$$\left|\frac{\partial L}{\partial w}\right| \le M\cdot\frac{1}{|z-2|}\left|\int_{-2}^{2}\int_{-2}^{2}\frac{4-xy}{\sqrt{4-x^2}\sqrt{4-y^2}}\,dx\,dy\right|.$$

$$\int_{-2}^{2}\int_{-2}^{2}\frac{4-xy}{\sqrt{4-x^2}\sqrt{4-y^2}}\,dx\,dy = 4\left(\int_{-2}^{2}\frac{1}{\sqrt{4-x^2}}\,dx\right)^2 - \left(\int_{-2}^{2}\frac{y}{\sqrt{4-y^2}}\,dy\right)$$

$$= 4\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1\,dt\right)^2 = 4\pi^2 \qquad (x = 2\sin t).$$

So, $\left|\frac{\partial L}{\partial w}\right| \le \frac{4\pi^2 M}{|z-2|}$, $\lim_{z\to\infty}\frac{\partial L}{\partial w} = 0$. $\lim_{z\to\infty}\frac{\partial S}{\partial w} = \lim_{z\to\infty}\frac{2\pi^2}{R(w)}\left(1 - \frac{\frac{R(z)}{z} - \frac{R(w)}{z}}{1-\frac{w}{2}}\right) = 0$.

$f(w) = \lim_{z\to\infty} f(w) = \lim_{z\to\infty}\frac{\partial S}{\partial w} - \frac{\partial L}{\partial w} = 0 - 0 = 0$. So, $f(w) = 0$.

$\Rightarrow \frac{\partial S}{\partial w} = \frac{\partial L}{\partial w} \Rightarrow S = L + g(z)$ for some analytic g.

By the same way as above, fix z and $w \to \infty$ for $w \in \mathbb{R}$.

Then $\left|\frac{\log(z-x) - \log(z-y)}{x-y}\right| = \left|\frac{1}{z - c(x,y,z)}\right| < M$ for some M,

$\left|\frac{\log(w-x) - \log(w-y)}{x-y}\right| \le \left|\frac{1}{w-2}\right|$, $|L| \le M\cdot\frac{1}{|w-2|}\cdot 4\pi^2$, So $\lim_{w\to\infty} L = 0$.

$$\lim_{w\to\infty} S = \lim_{w\to\infty} 2\pi^2 \log\left(\frac{(z+R(z))\left(1+\frac{R(w)}{w}\right)}{2\left(z - \frac{4}{w} + R(z)\cdot\frac{R(w)}{w}\right)}\right) = 2\pi^2\log 1 = 0.$$

So $g(z) = \lim_{w\to\infty} S - L = 0$. So $S = L$.

$$\therefore \int_{-2}^{2}\int_{-2}^{2}\left(\log(z-x) - \log(z-y)\right)\left(\log(w-x) - \log(w-y)\right)\frac{4-xy}{(x-y)^2\sqrt{4-x^2}\sqrt{4-y^2}}\,dx\,dy$$

$$= 2\pi^2\log\left(\frac{(z+R(z))(w+R(w))}{2(zw-4+R(z)R(w))}\right).$$