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The problem is equivalent to finding a pair (a, b) such that

$$ab | a^2 + b^2 + a + b + 1$$

in other words, we shall find a triple (a, b, k) of positive integers such that

$$kab = a^2 + b^2 + a + b + 1 \tag{1}$$

Firstly, we shall prove that $k = 5$. For a fixed k, let (a_0, b_0) be a solution where $a_0 + b_0$ is minimal. (If there are more than one, choose an arbitrary one.) Suppose that $a_0 \le b_0$. Consider the following quadratic equaion in t.

$$t^2 - (ka_0 - 1)t + a_0^2 + a_0 + 1 = 0$$

If t is a solution to the equation, then (a_0, t, k) is a solution for (1). One solution is $t = b_0$, as we have assumed. By Vieta's formula, the other solution is

$$t = b_1 = ka_0 - 1 - b_0 = \frac{a_0^2 + a_0 + 1}{b_0}$$

The first expression shows that this number is an integer, and the second expression shows that this number is positive. Therefore, (a_0, b_1, k) is also a valid solution for (1). By the minimality of $a_0 + b_0$, we must have

$$\frac{a_0^2 + a_0 + 1}{b_0} \ge b_0 \quad \Longrightarrow \quad a_0 \le b_0$$

since the successive square to a_0^2 is $a_0^2 + 2a_0 + 1$. Therefore, $a_0 = b_0$. But then,

$$k = \frac{2a_0^2 + a_0 + 1}{a_0^2}$$

and since a_0 is coprime with the numerator, we must have $a_0 = 1$, so $k = 5$.

Next we find all solutions to $a^2 + b^2 + a + b + 1 = 5ab$. If we have a solution (a, b) with $a \le b$, following the above paragraph, we get another solution (b', a) with $b' = 5a - b - 1$. We also have $b' < b$ or otherwise following the same argument as above will lead to $a = b$ and thus $(a, b) = (1, 1)$. Since $b' + a < a + b$, we eventually reach the minimal solution which was $(1, 1)$. But then, going backward is also nothing but Vieta's formula, so all solutions can be obtained by

$$(b, a) \to (a.5a - b - 1)$$

starting from $(1, 1)$. And of course, if (a, b) is a solution, (b, a) is also a solution.