# POW 2016-13

Lee, Jongwon

Define a sequence $\{s_n\}_{n\in\mathbb{N}}$ as follows

$$s_1 = 2$$
$$s_{n+1} = 1 + \prod_{i=1}^{n} s_i$$

Then, by an easy induction, one can prove that

$$\sum_{i=1}^{n} \frac{1}{s_i} = 1 - \frac{1}{s_1 s_2 \cdots s_n}$$

for all $n$.

We wish to show that if $k$ positive integers $a_1 \le a_2 \le \cdots \le a_k$ satisfy

$$\sum_{i=1}^{k} \frac{1}{a_i} < 1$$

then we have

$$\sum_{i=1}^{k} \frac{1}{a_i} \le \sum_{i=1}^{k} \frac{1}{s_i}$$

We shall use mathematical induction on $k$. If $k = 1$, it is trivial.

Now suppose the statement holds for all numbers less than $k$. Assume, for the sake of contradiction, that

$$\sum_{i=1}^{k} \frac{1}{a_i} > \sum_{i=1}^{k} \frac{1}{s_i}$$

Then, the Abel summation formula gives

$$\sum_{i=1}^{k} a_i \left(\frac{1}{s_i} - \frac{1}{a_i}\right) = \sum_{i=1}^{k-1} (a_i - a_{i+1}) \left(\sum_{j=1}^{i} \left(\frac{1}{s_j} - \frac{1}{a_j}\right)\right) + a_k \left(\sum_{j=1}^{k} \left(\frac{1}{s_j} - \frac{1}{a_j}\right)\right) \le 0$$

by the assumptions and the induction hypotheses. Therefore, we have

$$\sum_{i=1}^{k} \frac{a_i}{s_i} \le k$$

and the application of AM-GM inequality on the LHS gives

$$a_1 a_2 \cdots a_k \le s_1 s_2 \cdots s_k$$

But then, since $(1/a_1) + \cdots + (1/a_k)$ can be represented with a fraction with $a_1 a_2 \cdots a_k$ as the denominator,

$$\sum_{i=1}^{k} \frac{1}{a_i} \le \frac{a_1 a_2 \cdots a_k - 1}{a_1 a_2 \cdots a_k} \le \frac{s_1 s_2 \cdots s_k - 1}{s_1 s_2 \cdots s_k} = \sum_{i=1}^{k} \frac{1}{s_i}$$

which is a contradiction. Therefore, we have proved the desired statement.

The original problem is equivalent to

$$\frac{1}{N_1} + \cdots + \frac{1}{N_k} = 1 - \frac{1}{N+1}$$

Then, by the result of the above, we have

$$1 - \frac{1}{N+1} \le \sum_{i=1}^{k} \frac{1}{s_i} = 1 - \frac{1}{s_1 s_2 \cdots s_k} \quad \Longrightarrow \quad N \le s_1 s_2 \cdots s_k - 1$$

and this maximum can actually by obtained by letting $N_i = s_i$ for all $i$. In this case, we can also check that $N_i = s_i$ divides $N + 1 = s_1 \cdots s_k$ for all $i$. Therefore, the answer is $s_1 \cdots s_k - 1 = s_{k+1} - 2$.