KAIST POW 2016-12

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Problem. *Let k be a positive integer. Let $a_n = 1$ if n is not a multiple of $k+1$, and $a_n = -k$ if n is a multiple of $k+1$. Compute*

$$\sum_{n=1}^{\infty} \frac{a_n}{n}.$$

Solution. Let $s_i := \sum_{n=1}^{i} a_n$. Note that for positive integer t, $s_{t(k+1)} = \sum_{n=1}^{t(k+1)} \frac{1}{n} - \sum_{i=1}^{t} \frac{(k+1)}{i(k+1)} = \sum_{n=t+1}^{t(k+1)} \frac{1}{n}$, and

$$\left|\ln(k+1) - s_{t(k+1)}\right| = \left|\left(\int_t^{t(k+1)} \frac{1}{x}\,dx\right) - s_{t(k+1)}\right| \le \sum_{n=t}^{t(k+1)} \left|\left(\int_n^{n+1} \frac{1}{x}\,dx\right) - \frac{1}{n+1}\right|$$

$$\le \sum_{n=t}^{t(k+1)} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{1}{t} - \frac{1}{t(k+1)+1} \le \frac{1}{t}.$$

Because $1/t \to 0$ as $t \to \infty$, we have $\lim_{t\to\infty} s_{t(k+1)} = \ln(k+1)$.

Finally, for positive integer $i > k+1$, let M be an integer with $(k+1)M \le i < (k+1)(M+1)$. Then we have

$$s_{(k+1)M} \le s_i = s_{(k+1)M} + \sum_{(k+1)M+1}^{i} \frac{1}{t} \le s_{(k+1)M} + (k+1)\cdot\frac{1}{(k+1)M}.$$

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In other words, s_i is bounded by $s_{(k+1)\lfloor i/(k+1)\rfloor}$ and $s_{(k+1)\lfloor i/(k+1)\rfloor} + 1/\lfloor i/(k+1)\rfloor$. Because both sequence converges to $\ln(k+1)$, we have $\lim_{i\to\infty} s_i = \ln(k+1)$.