Pow 7

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Let's pick $r \in (0, T]$ uniformly, where T is sufficiently large.
Denote $\langle x \rangle$ as the fractional part of x.
Define $S_r := \{a \in A \mid \langle r \cdot a \rangle \in (\frac{1}{3}, \frac{2}{3}]\}$. For any $a \in A$, note that $\Pr[a \in S_r]$ could be close to $1/3$ by taking large T.
(i.e. $\forall \varepsilon > 0$, $\exists T > 0$ s.t. $\Pr[a \in S_r] > \frac{1}{3} - \varepsilon$). Write $A = \{a_1, \cdots, a_n\}$
where $|A| = n$. From above, $\exists T_0 > 0$ s.t. $\Pr[a_k \in S_r] > \frac{1}{3} - \frac{1}{6n}$, $\forall k \in [n]$.
Then, $E[|S_r|] = \sum_{a_k \in A} \Pr[a_k \in S_r] > \frac{n}{3} - \frac{1}{6}$, which implies that $\exists r \in (0, T]$
s.t. $|S_r| > \frac{n}{3} - \frac{1}{6}$. As $|S_r| \in \mathbb{N}$, we have $|S_r| \geq \frac{n}{3}$.
Now, assume that S_r is not sum-free, then $\exists a, b, c \in S_r$ s.t. $a + b = c$
$\Rightarrow ra + rb = rc \Rightarrow \langle ra + rb \rangle = \langle rc \rangle \Rightarrow \langle ra \rangle + \langle rb \rangle = \langle rc \rangle$ ... (*)
However, (*) is impossible by the def. of S_r, thus S_r is sum-free w/ $|S_r| \geq n/3$. From these, $f(A) \geq \frac{|A|}{3}$ for any A not containing 0, so
$f(n) = \min_{\substack{|A| = n \\ 0 \notin A}} f(A) \geq n/3$.