Pow 5 2015 국윤범

Suppose $A_1, A_2, \cdots, A_n$ are given. Here, assume $|A_j| \neq 0$ $\forall j \in [n]$. Now, define

$$A = (A_{ij}) : A_{ij} = \begin{cases} 1/|A_j| & i \in A_j \\ 0 & \text{o.w.} \end{cases}, \quad B = (B_{ij}) : B_{ij} = \frac{1}{n} \quad \text{and}$$

$M = (1-p)A + pB$ where p is a fixed value in $(0,1)$.

Observe that $\sum_{i=1}^{n} M_{ij} = 1$, $\forall j \in [n]$ and $M_{ij} > 0$ $\forall i,j \in [n]$. By Perron-Frobenius thm, 1 is an eigenvalue of M with an eigenvector $\begin{pmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \end{pmatrix}$ where all $T_i > 0$.

Define $\omega_{ij} := M_{ij} T_j$, $\forall i,j \in [n]$. Then, observe that

$$\sum_{i=1}^{n} \omega_{ij} = T_j \left(\sum_{i=1}^{n} M_{ij} \right) = T_j \ \& \ \sum_{i=1}^{n} \omega_{ji} = \sum_{i=1}^{n} M_{ji} T_i = T_j \quad (\because \begin{pmatrix} T_1 \\ \vdots \\ T_n \end{pmatrix} \text{ is an eigenvector for } 1) \quad ①$$

Now, take $p < \frac{1}{n}$, and define a partition of $[n]$, $\{C_1, C_2, C_3, C_4\}$, which minimizes $I := \sum_{k=1}^{4} \left(\sum_{i,j \in C_k} \omega_{ij} \right)$. Suppose that $\exists i \in C_1$ s.t. $|A_i \cap C_1| > \frac{1}{2}|A_i|$, then $|A_i \cap C_1| \geq \frac{1}{2}(|A_i|+1)$ $(*)$

Define $p_k := \sum_{j \in C_k} \omega_{ji}$ and $q_k := \sum_{j \in C_k} \omega_{ij}$, $\forall k \in [4]$. Then, observe the following:

$$p_1 = \sum_{j \in C_1} \omega_{ji} \geq \sum_{j \in C_1 \cap A_i} \omega_{ji} = \sum_{j \in C_1 \cap A_i} \left(p\frac{T_i}{n} + (1-p) A_{ji} T_i \right) > \sum_{j \in C_1 \cap A_i} (1-p) A_{ji} T_i$$

$$\geq (1-p) T_i \cdot \tfrac{1}{2}(|A_i|+1) / |A_i| \ (\text{by } (*)) > \frac{T_i}{2} \quad (\because p < \tfrac{1}{n} \ \& \ |A_i| \leq n-1)$$

By ①, $\sum_{k=1}^{4} p_k = \sum_{k=1}^{4} q_k = T_i$, so $p_1 + q_1 > \frac{T_i}{2} = \frac{1}{4} \sum_{k=1}^{4} (p_k + q_k)$. Thus $\exists t \in \{2,3,4\}$ s.t. $p_t + q_t < p_1 + q_1$. WLOG, let $t = 2$. If "i" were in C_2, then the I value would be $I' = I - (p_1 + q_1 - \omega_{ii}) + (p_2 + q_2 - \omega_{ii}) = I - (p_1 + q_1) + (p_2 + q_t) < I$. $\times$

Now, suppose, $\exists j \in [n]$ s.t. $|A_j| = 0$. In this case, add any elt in $[n]$ to A_j and construct a partition by the same way. Note that the addition of new elt to A_j with $|A_j| = 0$ doesn't matter at all if we ignore all redundant inequalities w.r.t. such A_j.