Pow 6

2015 측정법

Let A, B, M be Hermitian mtx, and $t \in [0, 1]$. Let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of M. Define $V_{\lambda_i} := \{v \in \mathbb{C}^n : Mv = \lambda_i v\}$ (i.e. eigenspace of λ_i), $\forall i \in [n]$.

By spectral decomposition, we can write $M = \sum_{i=1}^{n} \lambda_i P_{\lambda_i}$

(where P_{λ_i} is the orthogonal projection onto V_{λ_i})

For $f : \mathbb{R} \to \mathbb{R}$ convex, note that $f(M) = \sum_{i=1}^{n} f(\lambda_i) P_{\lambda_i}$. Now, say $\{u_1, \cdots, u_n\}$ is any orthonormal base of $\mathbb{C}^n$. Then observe the followings:

$$\left.\begin{aligned} \mathrm{Tr}(f(M)) &= \sum_{i=1}^{n} \langle u_i, f(M) u_i \rangle = \sum_{i=1}^{n} \langle u_i, (\sum_{j=1}^{n} f(\lambda_j) P_{\lambda_j}) u_i \rangle = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} f(\lambda_j) \| P_{\lambda_j} u_i \|^2 \right) \\ &\underset{(*)}{\geq} \sum_{i=1}^{n} f\left(\sum_{j=1}^{n} \lambda_j \| P_{\lambda_j} u_i \|^2 \right) = \sum_{i=1}^{n} f(\langle u_i, (\sum_{j=1}^{n} \lambda_j P_{\lambda_j}) u_i \rangle) = \sum_{i=1}^{n} f(\langle u_i, M u_i \rangle) \end{aligned}\right\} (\#)$$

((*): Since f is convex and $\sum_{j=1}^{n} \| P_{\lambda_j} u_i \|^2 = \| u_i \|^2 = 1$, this inequality holds)

Now, let $\{u_1, \cdots, u_n\}$ be an orthonormal basis of $\mathbb{C}^n$ derived from $tA + (1-t)B$.

$$\Rightarrow \mathrm{Tr}(f(tA + (1-t)B)) = \sum_{j=1}^{n} f(\langle u_j, (tA + (1-t)B) u_j \rangle) = \sum_{j=1}^{n} f(t\langle u_j, A u_j \rangle + (1-t) \langle u_j, B u_j \rangle)$$

$$\underset{\text{convex}}{\leq} \sum_{j=1}^{n} \left(t f(\langle u_j, A u_j \rangle) + (1-t) f(\langle u_j, B u_j \rangle) \right)$$

$$\underset{(\#)}{\leq} t \cdot \mathrm{Tr}(f(A)) + (1-t) \mathrm{Tr}(f(B))$$

Hence, the map $X \mapsto \mathrm{Tr}(f(X))$ is also convex.

<u>NOTE</u> The following are some facts that I used in the proof.

① $\mathrm{Tr} M = \sum_{i=1}^{n} \langle u_i, M u_i \rangle$ where $\{u_1, \cdots, u_n\}$ is any orthonormal base of $\mathbb{C}^n$.

② P: orthogonal projection, so $\langle x, Px \rangle = \langle Px, Px \rangle = \| Px \|^2$