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For a real number s such that $-\frac{1}{2} \leq s \leq 0$, call the following sequence of $\{\epsilon_i\}$, a s-good sequence.

Define p_i as $s + \sum_{k=1}^{j} \epsilon_i x_i$, and ofcourse, $p_0 = s$. Then, $\{\epsilon_i\}$ is called a s-good sequence if $\epsilon_i = \frac{1}{2}$ when $p_{i-1} < 0$ and $\epsilon_i = -\frac{1}{2}$ when $p_{i-1} > 0$. Then, we call p_n a s-exciting value. Also, it is obvious that $|p_i| \leq \frac{1}{2}$ for all s-good sequences $\{\epsilon_i\}$.

Note that an s-good sequence can be determined from $i = 1$ to n inductively, and uniquely if p_i never becomes zero. Therefore, for all but finitely many values s, s-exciting value is unique.

Now if $p_i = 0$ for some i, we have to choices for the sign of ϵ_{i+1}. To make ϵ_{i+1} the opposite sign, we can just flip the whole sign of ϵ_j for all $j > i$. Even in the case when there are more than one i such that $p_i = 0$, we can start flipping from the front making all ϵ_{i+1}'s the signs we want. But then, note that in this case, p_n only changes its sign, and s-exciting values are $\{x, -x\}$ for some x.

Suppose we are increasing s and observing s-exciting values. When p_i's are not zero for all i, the unique s-exciting value increases with slope 1 w.r.t. s. When s meets a point where $p_i = 0$ for some i's, then to continue increasing p_n with slope 1 w.r.t. s, we have to flip signs appropriately. Specifically, ϵ_{i+1} must be $-1/2$ for all i with $p_i = 0$. After fliiping signs appropriately, we can begin increasing s and p_n with the same slope.

With the observations we have made, we can conclude that $|p_n|$ is a continuous function of s, say $f(s)$, and it has slope ± 1 except for finitely many points, and a cusp when the slope changes. Also, for an open interval where $f(s)$ has slope 1, $f(s)$ is the unique s-exciting value, for an open interval where $f(s)$ has slope -1, $-f(s)$ is the unique s-exciting value, and in a cusp, $\pm f(s)$ are s-exciting values.

Now we show that there exists s such that $-s$ is a s-exciting value.

Since $f(-\frac{1}{2}) \leq \frac{1}{2}$ and $f(0) \geq 0$, by the intermediate value theorem, there exists $s_0 \in [-1/2, 0]$ such that $f(s_0) = |s_0| = -s_0$. If s_0 belongs to an open interval with slope 1, or if $f(s)$ is a cusp in $s = s_0$, then $-s_0$ is a s_0-exciting value as we wanted.

Now suppose s belongs to a maximal open interval (a, b) such that f has slope -1 in the interval. Then, $f(b)$ is a cusp or b must be 0. If $b = 0$, $f(0) = 0$, so that -0 is a 0-exciting value as we wanted. If $f(b)$ is a cusp, then $-b = f(b)$ is a b-exciting value.

Now if $\{\epsilon_i\}$ is a s-good sequence such that $p_n = -s$, then

$$\begin{aligned}
\left|\sum_{i=1}^{k} \epsilon_i x_i - \sum_{i=k+1}^{n} \epsilon_i x_i\right| &\leq \left|\sum_{i=1}^{k} \epsilon_i x_i\right| + \left|\sum_{i=k+1}^{n} \epsilon_i x_i\right| \\
&= |p_k - p_0| + |p_n - p_k| \\
&= |p_k - s| + |p_k + s| \\
&\leq \max\left(|\frac{1}{2} - s| + |\frac{1}{2} + s|, |-\frac{1}{2} - s| + |-\frac{1}{2} + s|\right) = 1
\end{aligned}$$

where the last inequality holds from the convexity of $|x - s| + |x + s|$.