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Let $v_1, \cdots, v_n$ be orthonormal eigenvectors of A with eigenvalues $\lambda_1, \cdots, \lambda_n$, and $w_1, \cdots, w_n$ be orthonormal eigenvectors of B with eigenvalues $\mu_1, \cdots, \mu_n$.

$$\mathrm{Tr}(AB) = \sum_{i=1}^{n} v_i^* A^* B v_i = \sum_{i=1}^{n} \lambda_i v_i^* B v_i = \sum_{j=1}^{n} \sum_{i=1}^{n} \lambda_i v_i^* B w_j w_j^* v_i = \sum_{j=1}^{n} \sum_{i=1}^{n} \lambda_i \mu_j |v_i^* w_j|^2.$$

By Hölder's inequality on a weighted (product) counting measure,

$$|\mathrm{Tr}(AB)| = \left| \sum_{j=1}^{n} \sum_{i=1}^{n} \lambda_i \mu_j |v_i^* w_j|^2 \right| \leq \left(\sum_{j=1}^{n} \sum_{i=1}^{n} |\lambda_i|^p |v_i^* w_j|^2 \right)^{1/p} \left(\sum_{j=1}^{n} \sum_{i=1}^{n} |\mu_j|^q |v_i^* w_j|^2 \right)^{1/q}$$

, and by the orthonormality of eigenbases,

$$|\mathrm{Tr}(AB)| \leq \left(\sum_{j=1}^{n} \sum_{i=1}^{n} |\lambda_i|^p |v_i^* w_j|^2 \right)^{1/p} \left(\sum_{j=1}^{n} \sum_{i=1}^{n} |\mu_j|^q |v_i^* w_j|^2 \right)^{1/q} = \left(\sum_{i=1}^{n} |\lambda_i|^p \right)^{1/p} \left(\sum_{j=1}^{n} |\mu_j|^q \right)^{1/q} = ||A||_{S^p} ||B||_{S^q}.$$

$\square$