POW 2015-20

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Let S be the set of real numbers in $[0,1)$ where the base-8 expansion consists of 1 and 4 except for finitely many digits. For an elements $x \in S$, let $x = 0.a_1 \dots a_k b_1 \dots$, where $b_i = 1, 4$ for all i and $a_k \neq 1, 4$. Then, define $g(x)$ to be $0.c_1 c_2 \dots$ where $c_i = 0$ if $\cancel{a_i} = 1$ and $c_i = 1$ if $\cancel{a_i} = 4$, and read in binary. For all $x \notin S$, let $g(x) = 0$.

b_i

First, we show that S has measure zero. Let T be the set of real numbers in $[0,1)$ such that the density of 1 in its binary expansion is 0.5. By the strong law of large numbers, $\mu(T) = 1$. Since the binary expansion of an element of S consists of many 001 and 100's, the density of 1 is 1/3, so that $S \subseteq [0,1] - T$. Therefore, S is a measure zero set.

Secondly, for any open interval I, we show that $g(I) = [0,1]$. Since I is open, there exists k, n such that $\left[\frac{k}{8^n}, \frac{k+1}{8^n}\right] \subseteq I$. Then, if $\frac{k}{8^n} = 0.a_1 a_2 \dots a_k$ in base-8 expansion, I contains all numbers having $a_1 a_2 \dots a_k$ as prefix in base-8 expansion. Especially, it contains all numbers having $a_1 \dots a_k 0$ as prefix. Then, varying the suffix, we can generate all binary expansions from 0 to 1 as the image of g.

Finally, if we compose g with a bijection from $[0,1]$ to $\mathbb{R}$, such that 0 maps to 0, and extend the domain to $\mathbb{R}$ giving period 1, then we get the desired function.