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Sol) We will construct such a function.

Let C be the Cantor set. It is well-known that Cantor set has same cardinality with $[0,1]$. Since $[0,1]$ has same cardinality with $R/\{0\}$, there exists a function g from C to $R/\{0\}$ that is surjective (in fact, bijective).

For $x \in C$, let $S_x = \{q+x \mid q \in Q\}$. We will check some properties of S_x in the following claim.

Claim 1) For $\forall x \in C$, S_x is countable and dense. For $a \neq b$ and a, $b \in S$, $S_a \cap S_b = \phi$ or $S_a = S_b$. Also, $x \in S_x$.

pf)

1) S_x is just a translation of Q, so it is countable and dense.

2) Assume that $t \in S_a \cap S_b$. $t = q_1 + a = q_2 + b$. Hence, $b - a = k \in Q$. For $\forall r \in S_a$, let $r = q_3 + a$. Then, $r = (q_3 - k) + (a + k) = (q_3 - k) + b \in S_b$ which means $S_a \subseteq S_b$. Similarly, $S_b \subseteq S_a$. Hence, $S_a = S_b$.

3) Trivial because $0 \in Q$.

By Claim 1-2, we can make a family of real numbers $F \subset C$ such that for $\forall a, b \in F$, $S_a = S_b$. Let S_F be a set such that $S_F = S_a$ for $a \in F$. For all family F, F's are pairwise disjoint and their union is C. Also S_F's are pairwise disjoint.

Before we construct a function f, consider another function.

Claim 2) There exists function $h_1 : Q \to N$ that satisfies the following property.

Property : For any nonempty open interval I and natural number n, there exists a rational number $t \in I$ such that $h_1(t) = n$.

pf) Let p_k be the kth smallest prime number. Let h_1 be a function such that $h_1(0) = 1$, and for a rational number $t = \dfrac{q}{p}$ ($p \in N$, $q \in Z/\{0\}$. p and q are relatively prime), $h_1(t) = k$ if p_k is the smallest prime number that divides p. Now we will show that h_1 meets the given condition. Let $I = (a, b)$ be an arbitrary open interval, and n a natural number. For sufficiently large $m \in N$, there exists an nonzero integer s that satisfies $p_n^m a < s < p_n^m b$. Then $h_1(t) = n$ for $t = \dfrac{s}{p_n^m} \in I$.

Now, we will construct a function f that satisfies the conditions. Map a real number t by the following cases.

Case 1) t is not an element of any of S_F

$f(t)=0$ (i.e. map t to zero by f).

Case 2) t is an element of S_F for the proper family F.

By Claim 1-3, for $\forall x \in F$, $x \in S_x = S_F$. Hence, $F \subset S_F$. Since S_F is countable by Claim 1-1, F is also countable (no matter it has finite or infinitely many elements). So there exists a surjective function $h_0 : N \rightarrow F$. Let h_2 be a translation function from S_F to Q. Let h_1 be a function from Claim 2. Let $h_F : S_F \rightarrow F$ be a function such that $h_F = h_0 \circ h_1 \circ h_2$, which is obviously surjective. Since S_F is a translation of Q and h_2 is a translation function, the Property of h_1 stated in Claim 2 can be applied to h_F.

Property : For any nonempty open interval I and $k \in F$, there exists a real number $l \in S_F \cap I$ such that $h_F(l) = k$.

Let $f(t) = (g \circ h_F)(t)$ (i.e. map t to $g(h_F(t))$ by f)

f is indeed a function since S_F's are pairwise disjoint. Now, we will show that f meets all the conditions given.

(1) $f \equiv 0$ almost everywhere.

pf) It is enough to show that $\bigcup_{x \in C} S_x$ is measure zero. Note that the Cantor set C is well-known to be measure zero. By the idea of double counting, it is easy to see that $\bigcup_{x \in C} S_x$ is a countable unions of translations of C ($\because Q$ is countable). Hence $\bigcup_{x \in C} S_x$ is measure zero, since measure is countably additive. So the first condition is proved.

(2) For any nonempty open interval I, $f(I) = R$

pf) Since $f \equiv 0$ almost everywhere, there exists $a \in I$ such that $f(a) = 0$. Now, let s be an arbitrary nonzero real number. Since $g : C \rightarrow R/\{0\}$ is surjective, there exists $c \in C$ such that $g(c) = s$. For the family F that contains c, consider the function $h_F : S_F \rightarrow F$ in Case 2. By the Property stated in Case 2, there exists $t \in S_F \cap I$ such that $h_F(t) = c$. For this t, $f(t) = (g \circ h_F)(t) = s$. So the second condition is proved.

Hence, f is a function that satisfies the given conditions.