POW 2015-20 : Dense function

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The statement is true. Define a function $f : \mathbf{R} \to \mathbf{R}$ as below:

$$f(x) := \begin{cases} L & \text{if } \lim_{n\to\infty} \tan n!\pi x \text{ exists and it equals to } L \\ 0 & \text{otherwise} \end{cases}$$

for every $x \in \mathbf{R}$. Then f satisfies following properties:

Proposition 1. $f = 0$ almost everywhere.

Proof. Since $f(z) = \tan \pi z$ is continuous at its domain with period 1 when it is injective within single period, existence of $\lim_{n\to\infty} \tan n!\pi x$ is equivalent to convergence of fractional value of $n!x$, but this sequence is equidistributed for almost every x(hence limit diverges), hence f equals zero for almost every x. We may need additional explanation for this 'almost everywhere' part.

Lemma 1. Let (a_n), $n = 1, 2, cdots$, be a given sequence of distinct integers. Then the sequence $\{a_n x\}$ is equidistributed for almost all real numbers x.

Proof. Well-known theorem. One proof can be given from Theorem 4.1, Uniform Distribution of Sequences (Wiley Interscience), by L. Kuipers and H. Niederreiter. □

□

Proposition 2. f is surjective in any nonempty open interval : in other words, for any given $a < b$ and c, we can find $x \in (a, b)$ that $f(x) = c$.

Proof. Before proving this statement, Let us prove "periodicity" of f.

Lemma 2. f is periodic for every rational period : in other words, $f(x + q) = f(x)$ for every $x \in \mathbf{R}$ and $q \in \mathbf{Q}$.

Proof. It is somewhat trivial : write $q = m/n$, then it is immediate to check that $\tan((n')!\pi x) = \tan((n')!\pi(x + q))$ for every $n' \geq 2n$, Hence x and $x + q$ shares existence of limit and value if it exists, which means they have equal function value. □

Hence, it remains to show that f is surjecttive, since we can put in any open interval by adding appropriate rational number.
Let $c \in \mathbf{R}$, then there exists $r \in [0,1)$ that $\tan(\pi r) = c$. Then define x by

$$x = \sum_{n=0}^{\infty} \frac{\lfloor nr \rfloor}{n!}.$$

since $0 \leq \lfloor nr \rfloor / n! \leq= 1/n!$ and $\sum_{n=0}^{\infty} 1/n! = e$ converges, we can verify that the series converges by using comparison method.
Write real sequence x_n, ϵ_n by

$$x_n = \sum_{k=0}^{n} \frac{\lfloor kr \rfloor}{k!}$$

$$\epsilon_n = x - x_n = \sum_{k=n+1}^{\infty} \frac{\lfloor kr \rfloor}{k!}.$$

Then, we have $n!x_n$ is an integer, hence $\tan(n!\pi x) = \tan(n!\pi\epsilon_n)$. Moreover,

$$n!\epsilon_n = \frac{\lfloor (n+1)r \rfloor}{n+1} + n! \sum_{k=n+2}^{\infty} \frac{\lfloor kr \rfloor}{k!}.$$

and we can verify that

$$\lim_{n\to\infty} \lfloor (n+1)r \rfloor n + 1 = r$$

and

$$\lim_{n\to\infty} n! \sum_{k=n+2}^{\infty} \frac{\lfloor kr \rfloor}{k!} = 0.$$

, where second statement holds since $n!\lfloor kr \rfloor / k! \leq 1/(n+1)1/(k-n-1)!$ for every $k \geq n+2$ while $\sum_{k=n+2}^{\infty} 1/(n+1)1/(k-n-1)! = (e-1)/(n+1) \to 0$ as $n \to \infty$. Therefore, we have $\lim_{n\to\infty} n!\pi\epsilon_n = \pi r$.
Hence, $\lim_{n\to\infty} \tan n!\pi x = \lim_{n\to\infty} \tan n!\pi\epsilon_n = \tan \pi r = c$ from continuity of tan, so $f(x) = c$. □

Hence f becomes example proving that the statement is true.