POW 2015-22

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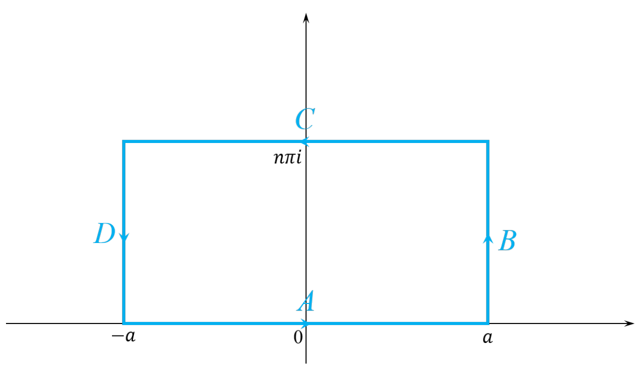
When $c = 0$, $\int_0^\infty 1/\cosh^2(x)dx = \tanh(\infty) - \tanh(0) = 1$. From now on, we assume $c > 0$ by taking $|c|$ if necessary.

We'll prove that

$$\int_0^\infty \frac{\cos(cx)}{\cosh^2 x} dx = \frac{c\pi/2}{\sinh(c\pi/2)}.$$

Consider the integration of $\dfrac{e^{cix}}{\cosh^2 x}$ along the rectangular contour R:



where $n \in \mathbb{N}$, $a > 0$. Because we have the power series

$$e^{cix} = e^{cir}\left(1 + ci(x-r) + \frac{(ci(x-r))^2}{2!} + \cdots\right)$$
$$\cosh^2 x = -\frac{1}{2}\left(\frac{2^2}{2!}(x-r)^2 + \frac{2^4}{4!}(x-r)^4 + \cdots\right)$$

for $r = (m + \frac{1}{2})\pi i$, the residues can be easily calculated,

$$\operatorname{Res}\left(\frac{e^{cix}}{\cosh^2 x}, (m + \frac{1}{2})\pi i\right) = -ce^{-c(m+\frac{1}{2})\pi} i.$$

So we have

$$\int_R \frac{e^{cix}}{\cosh^2 x} dx = 2\pi i \sum_{m=0}^{n-1} -ce^{-c(m+\frac{1}{2})\pi} i$$

If we send n to ∞, then RHS converges to $\dfrac{c\pi}{\sinh(c\pi/2)}$. LHS is

$$\int_A \frac{e^{cix}}{\cosh^2 x} dx + \int_C \frac{e^{cix}}{\cosh^2 x} dx + \int_{B+D} \frac{e^{cix}}{\cosh^2 x} dx$$

, and as n and a goes to ∞, the first term converges to $\int_{-\infty}^{\infty} e^{cix}/\cosh^2(x)dx$. The second term is bounded by

$$\left|\int_C \frac{e^{cix}}{\cosh^2 x}dx\right| \leq \int_{-a}^{a} \frac{e^{-c\pi n}}{\cosh^2 x}dx < e^{-c\pi n}$$

, so it vanishes as $n, a \to \infty$. The third term is bounded by

$$\left|\int_{B+D} \frac{e^{cix}}{\cosh^2 x}dx\right| \leq 2\int_0^n \frac{e^{-ct}}{e^a - e^{-a}}dt < \frac{2}{c(e^a - e^{-a})}$$

, and it also vanishes as $n, a \to \infty$. Therefore we conclude

$$\int_0^{\infty} \frac{\cos(cx)}{\cosh^2 x}dx = \frac{1}{2}\mathfrak{Re}\left(\int_{-\infty}^{\infty} \frac{e^{cix}}{\cosh^2 x}dx\right) = \frac{c\pi/2}{\sinh(c\pi/2)}.$$

□