POW 2015-20

Sunghyuk Park, '14

November 8, 2015

There is such a function.

Proof. One such function is the *Conway base 13 function*. It is a function $f : \mathbb{R} \to \mathbb{R}$ defined as follows[1]:

- If $x \in \mathbb{R}$, first write x as a tridecimal (that is, write it in base 13) using 13 'underlined digit symbols' $\underline{0}, \underline{1}, \underline{2}, \underline{3}, \underline{4}, \underline{5}, \underline{6}, \underline{7}, \underline{8}, \underline{9}, \underline{+}, \underline{-}, \underline{\cdot}$, each corresponding to $0, 1, \cdots, 12$ respectively. There should be no trailing $\underline{\cdot}$ recurring. We ignore the leading sign and the tridecimal point.
- If from some point onward, and for the first time, the tridecimal expansion consists of an underlined ordinary decimal number, r, say, then define $f(x) = r$.
- Otherwise, define $f(x) = 0$.

The points x where f is nonzero must have finitely many $\underline{\cdot}$ in tridecimal expansion. Given a fixed position of finitely many $\underline{\cdot}$s in a tridecimal expansion, the set of all x having $\underline{\cdot}$s exactly in the position has measure zero, following the same argument showing that the Cantor set has measure zero; its measure is smaller than $\left(\frac{12}{13}\right)^k$ for arbitrary large $k \in \mathbb{N}$, so it must be 0. Because all the possible positions of finitely many $\underline{\cdot}$s is countable, $f \equiv 0$ almost everywhere.

Also, f attains every value of $\mathbb{R}$ on every open interval. This can be easily checked, as for $r \in \mathbb{R}$ and an open interval I containing x, there is $y \in I$ obtained from the tridecimal expansion of x by substituting the digits after some point sufficiently right by $\underline{\cdot\cdot r}$ (where $\underline{r}$ denotes the tridecimal digits corresponding to the decimal expansion of r), which gives $f(y) = r$.

□

[1] https://en.wikipedia.org/wiki/Conway_base_13_function