POW 2015-19

Sunghyuk Park, '14

November 3, 2015

$$\sum_{k=0}^{n} 2^k \tan \frac{\pi}{4 \cdot 2^{n-k}} = \frac{\sin \frac{\pi}{2^{n+1}}}{1 - \cos \frac{\pi}{2^{n+1}}} = \cot \frac{\pi}{2^{n+2}}$$

Proof. We use the following two well-known identities:

$$\tan x = \sum_{m=1}^{\infty} \frac{(-1)^{m-1} 2^{2m} (2^{2m} - 1) B_{2m}}{(2m)!} x^{2m-1}, \quad |x| < \frac{\pi}{2} \tag{1}$$

$$\frac{x}{e^x - 1} = \sum_{m=0}^{\infty} B_m \frac{x^m}{m!} \tag{2}$$

where B_m are Bernoulli numbers.

Using (1), we have

$$\begin{aligned} \sum_{k=0}^{n} 2^k \tan \frac{\pi}{4 \cdot 2^{n-k}} &= \sum_{m=1}^{\infty} \frac{(-1)^{m-1} 2^{2m} (2^{2m} - 1) B_{2m}}{(2m)!} (\alpha^{2m-1} + 2 \cdot (2\alpha)^{2m-1} + \cdots 2^n (2\alpha)^{2m-1}) \\ &= \sum_{m=1}^{\infty} \frac{(-1)^{m-1} 2^{2m} B_{2m}}{(2m)!} (2^{2(n+1)m} - 1) \alpha^{2m-1} \\ &= \frac{1}{\alpha} \sum_{m=1}^{\infty} \left(\frac{(-1)^{m-1} B_{2m}}{(2m)!} \right) (\pi^{2m} - (2\alpha)^{2m}) \end{aligned} \tag{3}$$

where $\alpha = \dfrac{\pi}{4 \cdot 2^n}$.

From (2) and the fact that $B_0 = 1$,

$$\sum_{m=1}^{\infty} \frac{(-1)^{m-1} B_{2m}}{(2m)!} x^{2m} = \mathfrak{Re} \left(\frac{-ix}{e^{ix} - 1} \right) + 1$$

, so

$$(3) = -\frac{1}{\alpha} \mathfrak{Im} \frac{2\alpha}{e^{2i\alpha} - 1} = -2 \cdot \mathfrak{Im} \frac{1}{e^{2i\alpha} - 1} = \frac{2 \sin \frac{\pi}{2^{n+1}}}{|e^{i \frac{\pi}{2^{n+1}}} - 1|^2} = \frac{\sin \frac{\pi}{2^{n+1}}}{1 - \cos \frac{\pi}{2^{n+1}}}$$

$\square$