Problem (POW 2015-17)**.** *Let H be an $N \times N$ positive definite matrix and $G = H^{-1}$. Let H' be $(N-1)\times(N-1)$ matrix obtained by removing the N-th row and the column of H, i.e, $H'_{ij} = H_{ij}$ for any $i, j = 1, 2, \cdots, N-1$. Let $G' = (H')^{-1}$. Prove that*

$$G_{ij} - G'_{ij} = \frac{G_{iN}G_{Nj}}{G_{NN}}$$

Proof. δ_{ij} : Kronecker delta, $i, j = 0, 1, \cdots, n-1$

$$\begin{aligned}
&\sum_{k\le n-1} H'_{ik}\left(G_{kj} - \frac{G_{kN}G_{Nj}}{G_{NN}}\right) \\
&\qquad = \sum_{k\le n-1} H_{ik}G_{kj} - \frac{G_{Nj}}{G_{NN}} \sum_{k\le n-1} H_{ik}G_{kN} \\
&\qquad = (\delta_{ij} - H_{iN}G_{Nj}) - \frac{G_{Nj}}{G_{NN}}(\delta_{iN} - H_{iN}G_{NN}) \\
&\qquad = \delta_{ij} - \frac{G_{Nj}}{G_{NN}}\delta_{iN} \\
&\qquad = \delta_{ij}
\end{aligned}$$

Therefore,

$$G'_{ij} = ((H')^{-1})_{ij} = G_{ij} - \frac{G_{iN}G_{Nj}}{G_{NN}}$$

□