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Sol)

Let $D_n = \det(A_n)$. Using cofactor expansion to the first row, we obtain the following.

$$D_n = \det\left(\begin{bmatrix} 1 & x & 0 & \cdots & 0 \\ x & 1 & x & \cdots & 0 \\ 0 & x & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}\right) = 1 \cdot D_{n-1} - x \cdot \det\left(\begin{bmatrix} x & X \\ O & A_{n-2} \end{bmatrix}\right) = D_{n-1} - x^2 D_{n-2},$$

where X is an $1 \times (n-2)$ matrix $[x\,0\,0 \cdots 0]$, and O is an $(n-2) \times 1$ zero matrix.

It is easy to see the initial conditions $D_1 = 1$ and $D_2 = 1 - x^2$, so we have to solve the following recurrence relation : $D_{n+2} = D_{n+1} - x^2 D_n$ $(n \geqq 1)$, $D_1 = 1$, $D_2 = 1 - x^2$.

Its characteristic equation is $r^2 - r + x^2 = 0$.

i) $x = \pm \frac{1}{2}$

There is only one eigenvalue $\lambda = \frac{1}{2}$, so we can set $D_n = \left(\frac{1}{2}\right)^{n-1}(An+B)$. Based on two initial conditions, we obtain $D_n = \left(\frac{1}{2}\right)^n (n+1)$.

ii) $x \neq \pm \frac{1}{2}$

There are two eigenvalues λ_1 and λ_2, so $D_n = A\lambda_1^{n-1} + B\lambda_2^{n-1}$, where

$\lambda_1 = \frac{1 - \sqrt{1-4x^2}}{2}$, $\lambda_2 = \frac{1 + \sqrt{1-4x^2}}{2}$. Based on two initial conditions, we obtain

$A = \frac{\lambda_2 - (1-x^2)}{\lambda_2 - \lambda_1}$, $B = \frac{(1-x^2) - \lambda_1}{\lambda_2 - \lambda_1}$.

By i) and ii), we can get the final solution.

$$D_n = \begin{cases} \left(\frac{1}{2}\right)^n (n+1) & (x = \pm \frac{1}{2}) \\ \frac{\lambda_2 - (1-x^2)}{\lambda_2 - \lambda_1}\lambda_1^{n-1} + \frac{(1-x^2) - \lambda_1}{\lambda_2 - \lambda_1}\lambda_2^{n-1} & (x \neq \pm \frac{1}{2}) \end{cases}$$

where $\lambda_1 = \frac{1 - \sqrt{1-4x^2}}{2}$, $\lambda_2 = \frac{1 + \sqrt{1-4x^2}}{2}$.