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We will use the following identities in the solution. Firstly, it is an easy exercise of integration by parts in elementary calculus course that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{2n} x dx = \frac{\pi}{2^{2n}} \binom{2n}{n}$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{2n+1} x dx = 0$$

where the second identity holds since the function is odd. Also, let $C_n = \frac{1}{n+1}\binom{2n}{n}$ be the catalan sequence. Then its generating function is

$$c(x) = \sum_{i \geq 0} C_i x^i = \frac{1 - \sqrt{1-4x}}{2x}$$

. See here for the proof. Then,

$$\begin{aligned}
\sum_{i \geq 1} \frac{C_i}{i} x^i &= \int_0^x \frac{c(x) - 1}{x} dx \\
&= -\frac{1 - \sqrt{1-4x}}{2x} - \log\left(\frac{x(1+\sqrt{1-4x})}{1-\sqrt{1-4x}}\right) + 1
\end{aligned}$$

. Then, by substituting $x = 2\sin\theta$, if $|z| > 2$,

$$\begin{aligned}
\int_{-2}^{2} \log(z-x)\sqrt{4-x^2}dx &= \int \log z\sqrt{4-x^2}dx + \int \log\left(1 - \frac{x}{z}\right)\sqrt{4-x^2}dx \\
&= 2\pi \log z + \int \sum_{n=1}^{\infty} -\frac{1}{n}\left(\frac{x}{z}\right)^n \sqrt{4-x^2}dx \\
&= 2\pi \log z - \sum \int \frac{1}{2n}\left(\frac{x}{z}\right)^{2n} \sqrt{4-x^2}dx \\
&= 2\pi \log z - \sum \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2nz^{2n}} 2^{2n+2} \sin^{2n}\theta \cos^2\theta dx \\
&= 2\pi \log z - \sum \frac{2^{2n+2}}{2nz^{2n}} \left(\frac{1}{2^{2n}}\binom{2n}{n} - \frac{1}{2^{2n+2}}\binom{2n+2}{n+1}\right)\pi \\
&= 2\pi \log z - \sum \frac{1}{nz^{2n}} \left(\frac{1}{n+1}\binom{2n}{n}\right)\pi \\
&= \pi\left(2\log z + \frac{z(z-\sqrt{z^2-4})}{2} + \log\left(\frac{z+\sqrt{z^2-4}}{z^2(z-\sqrt{z^2-4})}\right) - 1\right) \\
&= \pi\left(\frac{z(z-\sqrt{z^2-4})}{2} + \log\left(\frac{z+\sqrt{z^2-4}}{z-\sqrt{z^2-4}}\right) - 1\right)
\end{aligned}$$

But then, it is clear by Morera's theorem that the answer is an analytic function of z, so the same result holds for $|z| \leq 2$.