KAIST POW 2015-14

2015[redacted] Inhyeok Choi

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Problem. *Find all positive integers n such that the following statement holds:*

> *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function that has a unique critical point c. If f has a local maximum at c, then $f(c)$ is an absolute maximum of f.*

Solution. (1) $n = 1$

Suppose that $f(d) > f(c)$ for some d. WLOG $d > c$ (otherwise consider $g = f(2c - x)$). f has a local maximum at c, so for some $c \in I = (a, b)$, $f(x) \le f(c)$ on I. If the equality holds on (c, b), that means $f'(x) = 0$ on (c, b); contradiction. Thus $f(t) < f(c)$ for some $t \in (c, b)$. Obviously $d > c$. Then, the minimum of f on $[c, d]$ is not attained at c nor d because $f(t) < f(c), f(d)$. Thus the minimum occurs in (c, d), which is another critical point; contradiction. Thus such d does not exists, $f(c)$ is an absolute maximum of f.

(2) $n \ge 2$ Let $f(x_1, \cdots, x_n) = -x_1^2 - \sum_{k=2}^n (1 - x_1)^3 x_k^2$. f has continuous partial derivatives so it is differentiable everywhere. Remark that

$$\frac{\partial f}{\partial x_i} = \begin{cases} -2x_1 + 3(1 - x_1)^2 \sum_{k=2}^n x_k^2 & i = 1 \\ -2x_i(1 - x_1)^2 & i \neq 1 \end{cases}.$$

We seek for the condition of $\nabla f(\mathbf{x}) = \mathbf{0}$. $\partial f / \partial x_i = 0$ $(i \neq 1)$ forces $x_i = 0$ or $x_1 = 1$. However if $x_1 = 1$, $\partial f / \partial x_i = -2 \neq 0$; thus $x_i = 0$ for $i \ge 2$, and then $\partial f / \partial x_1 = -2x_1 = 0 \Rightarrow x_1 = 0$. Putting $x_1 = \cdots = x_n = 0$, we assure that $\mathbf{0}$ is the only critical point.

Moreover, for $0 < |\mathbf{x}| < 1$, $f(\mathbf{x}) \le f(\mathbf{0}) = 0$ because all of $-x_1^2, -(1 - x_1)^3 x_k^2$ are non-positive. Thus f has a local maximum at $\mathbf{0}$. However, remark $f(11, 1, \cdots, 1) = -121 + 1000(n - 1) > 0 = f(\mathbf{0})$ so $f(\mathbf{0})$ is not an absolute maximum. In conclusion, the statement holds for $n = 1$ only. ■