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Sunghyuk Park, 2014[redacted]

There is such a sequence.

Proof. Let c_n be the least common multiple of $\{1, \dots n\}$. It suffices to show that there is an integer sequence such that every integer appears infinitely many times, and the sequence is periodic modulo c_m for every positive integer m. This is because if a sequence is periodic modulo c_m, then it is also periodic modulo m.

Construct such a sequence inductively as follows:

Step1: $(\mathbf{1}) = 0$

Step2: $(\mathbf{2}) = \underline{0}, 1, \underline{0}, -1$

Step3: $(\mathbf{3}) = \underline{0, 1, 0, -1}$, 2, 3, 2, 1, 4, 5, 4, 3, $\underline{0, 1, 0, -1}$, -2, -1, -2, -3, -4, -3, -4, -5

$\vdots$

Step n : $(\mathbf{n})$

Step $n+1$: $(\mathbf{n{+}1}) = \underline{(\mathbf{n})}, c_n{+}(\mathbf{n}), 2c_n{+}(\mathbf{n}), \dots, c_{n+1}{-}c_n{+}(\mathbf{n}), \underline{(\mathbf{n})}, -c_n{+}(\mathbf{n}), -2c_n{+}(\mathbf{n}), \dots, -c_{n+1} + c_n + (\mathbf{n})$

(Here, $(\mathbf{n})$ denote the finite sequence in the n-th step, and $k + (\mathbf{n})$ is a translation of $(\mathbf{n})$ by k.)

In $(n+1)$-th step, the finite sequence is constructed using the n-th step in such a way that (i) the n-th step is preserved in front, (ii) it contains every integers x such that $|x| \leq c_{n+1} - 1$, (iii) it is just a finite concatenation of the n-th step when taken modulo c_n, and (iv) the exact n-th step appears at least twice in the $(n+1)$-th step.

Because of the construction, the sequence is periodic modulo m for all c_m, and every integer appears infinitely many times as an integer n appears at least 2^k times in the $(n+k+1)$-th step of the construction. □