POW 2015-9

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Fix N. If there is a solution $(a_0, \dots, a_n)$ with at least one of them 0, then the denominator becomes 1, and N is automatically the sum of n squares of integers. Therefore, we can assume that all solutions do not contain 0. Similarly, we can assume that $N > n$, since all integers less than n can be represented as the sum of n 1's and 0's.

Now take a solution where $a_0 + a_1 + \dots + a_n$ is minimal, say $(b_0, b_1, \dots, b_n)$. Without loss of generality, suppose that $b_n \geq b_{n-1} \geq \dots \geq b_0 \geq 1$. Consider the following equation in x.

$$x^2 - Nb_0b_1 \dots b_{n-1}x + (b_0^2 + \dots + b_{n-1}^2 - N) = 0$$

This is just a reformulation of our original equation, so $x = b_n$ is a solution. Then, if $x = x_0$ is another solution, by Vieta's formula, we must have

$$x_0 = Nb_0b_1 \dots b_{n-1} - b_n = \frac{b_0^2 + \dots + b_{n-1}^2 - N}{b_n}$$

We can see that x_0 is an integer. Also, $x_0 \neq 0$ by our assumption. If $x_0 < 0$, then we have $b_n \geq Nb_0b_1 \dots b_{n-1} + 1$, so that

$$\begin{aligned} &b_n^2 - Nb_0b_1 \dots b_{n-1}b_n + (b_0^2 + \dots + b_{n-1}^2 - N) \\ &\geq Nb_0b_1 \dots b_{n-1} + 1 + (b_0^2 + \dots + b_{n-1}^2 - N) \geq 1 \end{aligned}$$

which is a contradiction. Therefore, $x_0 > 0$ so that $(b_0, \dots, b_{n-1}, x_0)$ is another solution to our original equation. By the minimality of our solution, we must have $x_0 \geq b_n$. Therefore, we have

$$b_0^2 + \dots b_{n-1}^2 - N \geq b_n^2$$

But then,

$$\begin{aligned} N(1 + b_0 \dots b_n) &= b_0^2 + \dots b_{n-1}^2 + b_n^2 \\ &\leq 2(b_0^2 + \dots + b_{n-1}^2) - N \\ &\leq 2nb_{n-1}^2 - N \end{aligned}$$

so that $nb_0 \dots b_{n-2}b_{n-1}^2 < Nb_0 \dots b_n < 2nb_{n-1}^2$, or $b_0 \dots b_{n-2} < 2$. Therefore, we must have $b_0 = \dots = b_{n-2} = 1$, but then

$$b_{n-1}^2 > n - 1 + b_{n-1}^2 - N \geq b_n^2$$

is a contradiction. It seems that we should be careful about the case $n = 1$, but a contradiction arises from

$$b_0^2 > b_0^2 - N \geq b_1^2$$

finishing the whole proof.