POW 2015-7: Binomial Identity

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Prove or disprove that

$$\sum_{i=0}^{r}(-1)^i\binom{i+k}{k}\binom{n}{r-i}=\binom{n-k-1}{r}$$

if k, r are non-negative integers and $0 \le r \le n-k-1$.

Solution. The identity holds indeed. By Leibniz Rule, we have

$$\binom{n-k-1}{r}=\frac{1}{r!}\left[\frac{d^r}{dx^r}(x+1)^{n-k-1}\right]_{x=0}=\left[\sum_{i=0}^{r}\frac{1}{i!(r-i)!}\left(\frac{d^i}{dx^i}\frac{1}{(x+1)^{k+1}}\right)\left(\frac{d^{r-i}}{dx^{r-i}}(x+1)^n\right)\right]_{x=0}$$

$$=\left[\sum_{i=0}^{r}\left((-1)^i\frac{(i+k)!}{i!k!}\frac{1}{(x+1)^{k+i+1}}\right)\left(\frac{n!}{(r-i)!(n-r+i)!}(x+1)^{n-r+i}\right)\right]_{x=0}=\sum_{i=0}^{r}(-1)^i\binom{i+k}{k}\binom{n}{r-i}.$$

□