POW 2015-06

[redacted] 김기현

Problem. Let A be an unbounded subset of $\mathbb{R}$. Let T be the set of all real numbers t such that $\{tx - \lfloor tx \rfloor : x \in A\}$ is dense in $[0,1]$. Is T dense in $\mathbb{R}$?

Proof. We will show that T is dense in $\mathbb{R}$. Define $f_x(t) = tx - \lfloor tx \rfloor$ for $x \in A$. Define $T_{r,s} = \{t \in \mathbb{R} : f_x(t) \in (r,s) \text{ for some } x \in A\}$ for $r,s \in \mathbb{Q}$ with $0 \le r < s \le 1$. Observe the followings for such r, s:

- $T_{r,s}$ is open because
$$T_{r,s} = \bigcup_{x \in A} f_x^{-1}\big((r,s)\big)$$
where each $f_x^{-1}\big((r,s)\big)$ is nonempty open for all nonzero $x \in A$ and $f_0^{-1}\big((r,s)\big) = \emptyset$.
- $T_{r,s}$ is dense. Take any $t \in \mathbb{R}$ and $\varepsilon > 0$. Since A is unbounded, there exists $x \in A$ such that $|x| > 1/\varepsilon$. Then, $f_x\big((t-\varepsilon, t+\varepsilon)\big) = [0,1) \supseteq (r,s)$, so $T_{r,s} \cap (t-\varepsilon, t+\varepsilon) \neq \emptyset$.

As $\mathbb{R}$ is a complete metric space, $\mathbb{R}$ is a Baire space. Thus $T = \bigcap_{r,s} T_{r,s}$ is dense in $\mathbb{R}$. $\square$