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Problem. Let $M = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ be a positive semidefinite Hermitian matrix. Prove that

$$\text{rank } M \leq \text{rank } A + \text{rank } C$$

(Here, A, B, C are matrices.)

Solution. By definition of block matrix and adjoint, A and C are square matrices. Since M is a positive semidefinite Hermitian matrix, it has a Cholesky decomposition $M = LL^*$ where L is a lower triangular matrix denoted as $L = \begin{pmatrix} L_A & O \\ L_B & L_C \end{pmatrix}$, with same block structure of M. Then, $L_A L_A^* = A$ and $\begin{pmatrix} L_B & L_C \end{pmatrix} \begin{pmatrix} L_B & L_C \end{pmatrix}^* = \begin{pmatrix} L_B & L_C \end{pmatrix} \begin{pmatrix} L_B^* \\ L_C^* \end{pmatrix} = C$. Now, since rank $K =$ rank KK^* for every matrix K over $\mathbb{C}$, thus

$$\begin{aligned} \text{rank } M = \text{rank } L &\leq \text{rank } \begin{pmatrix} L_A & O \end{pmatrix} + \text{rank } \begin{pmatrix} L_B & L_C \end{pmatrix} = \text{rank } L_A + \text{rank } \begin{pmatrix} L_B & L_C \end{pmatrix} \\ &= \text{rank } L_A L_A^* + \text{rank } \begin{pmatrix} L_B & L_C \end{pmatrix} \begin{pmatrix} L_B & L_C \end{pmatrix}^* = \text{rank } A + \text{rank } C \end{aligned}$$

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