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We begin with an simple lemma.

Lemma. *Let A be a positive semidefinite hermitian matrix. Then $Ax = 0$ iff $x^*Ax = 0$ for vectors x.*

Proof. The forward implication is trivial. For the backward implication, suppose $x^*Ax = 0$. Let $A = L^*L$ be the Cholesky decomposition of A. Then,

$$|Lx|^2 = x^*L^*Lx = x^*Ax = 0$$

so we have $Lx = 0$, so that $Ax = L^*Lx = 0$. □

Back to our main problem,

Solution Let B be a $m \times n$ matrix, then A is a $m \times m$ matrix and C is a $n \times n$ matrix. Let $v \in \ker(A)$, then

$$\begin{bmatrix} v \\ 0_n \end{bmatrix}^* \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \begin{bmatrix} v \\ 0_n \end{bmatrix} = \begin{bmatrix} v \\ 0_n \end{bmatrix}^* \begin{bmatrix} Av \\ B^*v \end{bmatrix} = v^*Av = 0$$

so by the lemma, we have $\begin{bmatrix} v \\ 0_n \end{bmatrix} \in \ker(M)$. Similarly, if $v \in \ker(C)$, then

$$\begin{bmatrix} 0_m \\ v \end{bmatrix}^* \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \begin{bmatrix} 0_m \\ v \end{bmatrix} = v^*Cv = 0$$

so that $\begin{bmatrix} 0_m \\ v \end{bmatrix} \in \ker(M)$.

Since $X = \left\{ \begin{bmatrix} v \\ 0_n \end{bmatrix} | v \in \ker(A) \right\}$ and $Y = \left\{ \begin{bmatrix} 0_m \\ v \end{bmatrix} | v \in \ker(C) \right\}$ have trivial intersection, we have

$$\text{nullity}(M) \geq \dim(\text{span}(X \cup Y)) = \dim(X) + \dim(Y) = \text{nullity}(A) + \text{nullity}(C)$$

Now the problem statement follows from the rank nullity theorem.