KAIST POW 2015-02

Problem.

Let T be a triangle. Prove that if every point of a plane is colored by Red, Blue, or Green, then there is a triangle similar to T such that all vertices of this triangle have the same color.

Proof.

We'll use the following well-known theorem;

Theorem (Van der Waerden's theorem)**.** *Let r and k any positive integers. Then there exists a positive integer $W(r,k)$ such that if we color the integers $\{1, 2, \cdots, W(r,k)\}$ using r colors, there exist k integers in arithmetic progression all of the same color. (see here)*

From the theorem, we directly obtain the following corollary;

Corollary. *Let $n = W(r,k)$. Choose any collinear points $P_1, P_2, \cdots, P_n$ from a plane which satisfy $\overline{P_1P_2} = \overline{P_2P_3} = \cdots = \overline{P_{n-1}P_n}$ (i.e evenly spaced). Color the points using r colors. Then, there exist k points P_{i_1}, P_{i_2}, $\cdots$, A_{i_k} such that all the points have the same color and $\overline{P_{i_1}P_{i_2}} = \overline{P_{i_2}P_{i_3}} = \cdots = \overline{P_{i_{k-1}}P_{i_k}}$.* $\cdots(*)$

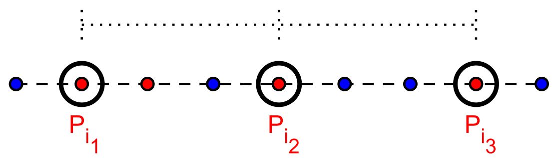

Figure 1: Example - when $r = 2$, $k = 3$ and $n = W(2,3) = 9$

Now assume that there is a 3-coloring(Red, Blue or Green) of the points on the plane such that there does NOT exist any monochromatic triangle similar to T. Let $s = W(2,3)$ and $t = W(3, s+1)$, and let $c(P) =$ (color of the point P).

Let's choose any t points on the plane which are collinear and evenly spaced. By $(*)$, there exists $(s+1)$ points $A_1, \cdots, A_{s+1}$ such that $c(A_1) = \cdots = c(A_{s+1})$ and $\overline{A_1A_2} = \cdots = \overline{A_sA_{s+1}}$. Without loss of generality, suppose that A_is are colored with Red.

Now define B_i as a point in the upper side of $\overline{A_1 A_{s+1}}$ such that $\triangle A_i B_i A_{i+1}$ is similar to T. (See figure 2)

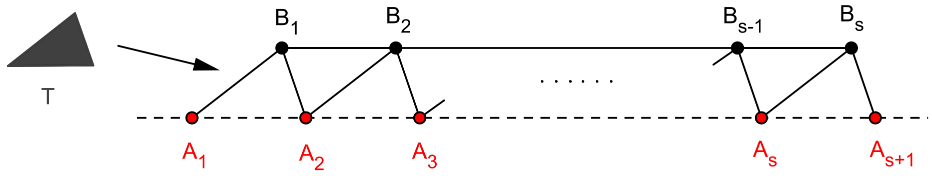

Figure 2: Points $B_1, \cdots, B_s$

If $c(B_i)$ is Red, then $\triangle A_i B_i A_{i+1}$ becomes monochromatic. So, $B_1, \cdots, B_s$ should be colored with Blue or Green, not Red. Applying (∗) again, there exist 3 points B_u, B_v, B_w such that $c(B_u) = c(B_v) = c(B_w)$ and B_v is the midpoint of B_u and B_w. Without loss of generality, suppose that those points are colored with Blue.

Now let's draw three points C, D, E such that $\triangle B_u C B_v$, $\triangle B_v D B_w$, and $\triangle CED$ are similar to T. (See figure 3)

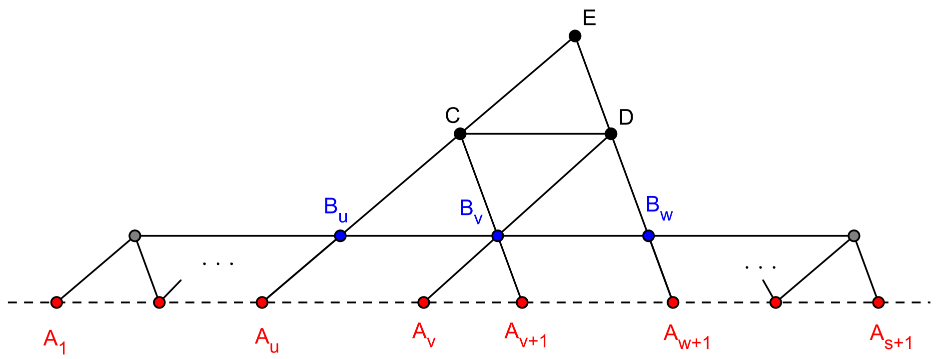

Figure 3: Points C, D, and E

C and D should be colored with Green; if we color one of them with Blue, one of the triangles $\triangle B_u C B_v$, $\triangle B_v D B_w$ will be Blue - monochromatic. Also, if we color one of them with Red, one the triangles $\triangle A_u C A_{v+1}$, $\triangle A_v D A_{w+1}$ will be Red - monochromatic.

However, we can't color E with any color! If E is colored with Red, Blue, or Green, then $\triangle A_u E A_{w+1}$, $\triangle B_u E B_w$, $\triangle CED$ will be Red, Blue, Green -

monochromatic respectively. It's contradiction.

Therefore, for any 3-coloring of the points in $\mathbb{R}^2$, there exists a monochromatic triangle similar to T. $\square$