No

Date

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Define $S_n = \sum_{i=1}^{n} a_i^5$.

$\{S_n\}_{n=1}^{\infty}$ is an increasing sequence, and $\lim_{n\to\infty} a_n S_n = 1$.

If $\{S_n\}$ converges, then $\{a_n\}$ also converges and it contradicts.

$\Rightarrow \lim_{n\to\infty} S_n = \infty$, $\lim_{n\to\infty} a_n = 0$.

$$\lim_{n\to\infty} \frac{S_{n+1}}{S_n} = 1 + \lim_{n\to\infty} \frac{S_{n+1}-S_n}{S_n} = 1 + \lim_{n\to\infty} \frac{a_{n+1}^5}{S_n} = 1.$$

$$\lim_{n\to\infty} (S_{n+1}^6 - S_n^6) = \lim_{n\to\infty} (S_{n+1}^5 + S_{n+1}^4 S_n + \cdots + S_n^5)\, a_{n+1}^5$$

$$= \lim_{n\to\infty} \left(1 + \frac{S_n}{S_{n+1}} + \cdots + \left(\frac{S_n}{S_{n+1}}\right)^5\right) S_{n+1}^5 a_{n+1}^5$$

$$= 6.$$

By ⟨lemma⟩ $\lim_{n\to\infty} \frac{S_n^6}{6n} = 1$. $\Rightarrow \lim_{n\to\infty} S_n (6n)^{-1/6} = 1$.

Then $\lim_{n\to\infty} a_n (6n)^{1/6} = \lim_{n\to\infty} a_n S_n = 1$.

⟨lemma⟩ $\{t_n\}$ is a convergent sequence of real number.

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} t_i.$$

(I guess I don't need to prove it ?!!)