POW 2014-23

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November 22, 2014

Theorem 1. *Let $f : [0,1] \to \mathbb{R}$ be a differentiable function with $f(0) = 0, f(1) = 1$. For every positive interge n, there exist $k_1, k_2, ..., k_n \in (0,1)$ such that $f(k_i) = i/n$ and $k_1 < k_2 < ... < k_n$.*

Proof. f is differentiable on $(0,1)$, so it is continuous on $[0,1]$. Because of intermediate value theorem, for every $i = 1, 2, ..., n$, there exist $k \in (0,1)$ such that $f(k) = i/n$. Let k_i be the smallest value of $k \in (0,1)$ such that $f(k) = i/n$.

$f(0) = 0 < i/n < (i+1)/n = f(k_{i+1})$. Therefore, there exists $t \in (0, k_{i+1})$ such that $f(t) = i/n$. k_i is the smallest value of $k \in (0,1)$ such that $f(k) = i/n$, so $k_i < k_{i+1}$. □

Now, we can prove the problem.

Problem 1. *Let $f : [0,1] \to \mathbb{R}$ be a differentiable function with $f(0) = 0, f(1) = 1$. Prove that for every positive interge n, there exist n distinct numbers $x_1, x_2, ..., x_n \in (0,1)$ such that $\frac{1}{n}\sum_{i=1}^{n} \frac{1}{f'(x_i)} = 1$*

Proof. Because of Theorem 1, there exist $k_1, k_2, ..., k_n \in (0,1)$ such that $f(k_i) = i/n$ and $k_1 < k_2 < ... < k_n$. Additionally, let $k_0 = 0$.

Use mean value theorem. f is differentiable on $(0,1)$, and it is continuous on $[0,1]$. Therefore, for every $i = 1, 2, 3, ..., n$, there exist $x_i \in (k_{i-1}, k_i)$ such that $f'(x_i) = \frac{f(k_i) - f(k_{i-1})}{k_i - k_{i-1}} = \frac{1/n}{k_i - k_{i-1}}$.

Now, $\frac{1}{n}\sum_{i=1}^{n} \frac{1}{f'(x_i)} = \frac{1}{n}\sum_{i=1}^{n} \frac{k_i - k_{i-1}}{1/n} = \sum_{i=1}^{n}(k_i - k_{i-1}) = k_n - k_0 = 1.$ □