

POW 2014 - 24

2013xxxx ~~July~~

Q. Suppose that n points are chosen randomly on a sphere. What is the probability that all points are on some hemisphere?

Sol) (i) Let $P(n)$ be the probability given in the question.

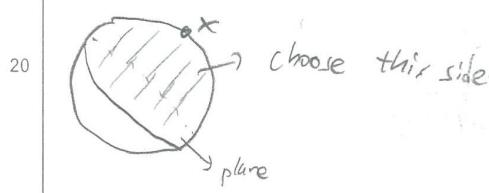
Instead of choosing n random points choose n random diameters of the sphere. And choose one of the two end points. Then except the case with zero probability measure, if we let $P'(n)$ be the probability that all points on some hemisphere when we choose n random diameters & their end points,

$P'(n)$ is independent to choice of diameters. $\therefore P(n) = P'(n)$

(ii) For each diameter chosen in (i), consider plane including the center of sphere & is perpendicular to this diameter.

Then we can find that these planes divide the sphere into many regions. Actually, each of them corresponds to a particular choice of endpoints that will make n points be on the same hemisphere.

I will show why. Choose a random point on sphere for each plane, (there will be 2 hemispheres each case),



choose the hemisphere which contains x . Then it is clear that choosing a plane gives a different choice of endpoints to get all points to be on the same hemisphere.

Thus the total number of the regions, $C_3(n)$, is proportional to $P'(n)$. Actually, there is 2^n choices of endpoints. Since there are n diameters & 2 endpoints for each diameter.

$$\text{thus } P'(n) = \frac{C_3(n)}{2^n}$$

(III) Define, $C_2(n)$, the regions of a circle in \mathbb{R}^2 divided by n lines, then obviously, $C_2(n) = 2^n$ (without cases with zero probability measure)

So let's find a relation between $C_2(n+1)$, $C_3(n)$, & $C_2(n)$

Consider a sphere where n planes already exist,

And new plane is added & slice some old regions into two smaller ones.

Then how many regions are sliced by the new plane?

Note that planes are 2-dimensional. Each of n old planes will intersect the new one along lines passing the center of the sphere. & the number of regions can be obtained by thinking adding n diameters on a circle which is 2-dimension version of original question & the number of cases is $C_2(n)$.

Again, thus $C_3(n)$ is equal to the number of regions sliced by a new plane, i.e. Adding a new plane increases the number of regions by $C_2(n)$

$$\therefore C_3(n+1) = C_2(n) + C_3(n)$$

$$\Rightarrow C_3(n) = \sum_{k=1}^{n-1} C_2(k) + C_3(1)$$

$$= n^2 - n + 2, \quad \begin{matrix} \text{clearly,} \\ (C_3(1)=2) \end{matrix}$$

by (I), (II), (III)

$$\Rightarrow P(n) = P'(n) = \frac{C_3(n)}{2^n} = \frac{n^2 - n + 2}{2^n}$$

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