Problem. *For a nonnegative real number x, let*

$$f(x) = \frac{\prod_{k=1}^{n-1}((x+k)(x+k+1))}{(n!)^2}$$

for a positive integer n. Determine $\lim_{n\to\infty} f_n(x)$.

Proof.

When $x=0$, $f_n(0)$ equals to $\frac{(n-1)!n!}{(n!)^2} = \frac{1}{n}$, which converges to 0.

For $x>0$, we use the fact that the following identity holds for $x>0$;

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!n^x}{x(x+1)\cdots(x+n)} \quad \cdots (*)$$

By $(*)$, we obtain

$$\begin{aligned} f_n(x) &= \frac{(x+1)(x+2)\cdots(x+(n-1))}{n!} \cdot \frac{(x+2)(x+3)\cdots(x+n)}{n!} \\ &\sim \frac{n^x}{x(x+n)\Gamma(x)} \cdot \frac{n^x}{x(x+1)\Gamma(x)} \qquad (\text{as } n\to\infty) \\ &\sim \frac{1}{x^2(x+1)\Gamma(x)^2} \cdot \frac{1}{1+\frac{x}{n}} \cdot n^{2x-1} \qquad (\text{as } n\to\infty) \end{aligned}$$

This shows that $f_n(x)$ is convergent only when $2x-1 \le 0$, and divergent otherwise.

Hence we obtain

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{when } 0 \le x \le \frac{1}{2} \\ \frac{1}{\left(\frac{1}{2}\right)^2(\frac{1}{2}+1)\Gamma\left(\frac{1}{2}\right)^2} = \frac{8}{3\pi} & \text{when } x = \frac{1}{2} \\ \infty & \text{when } x > \frac{1}{2} \end{cases}$$

□