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Note: $\mathbb{Z}^* = \mathbb{N} \cup \{0\}$. $\#(A)$ is the cardinality of A. $a \equiv_2 b \Leftrightarrow a \equiv b \pmod 2$.

Problem. *Let $\mathcal{F}$ be a non-empty collection of subsets of a finite set U. Let $D(\mathcal{F})$ be the collection of subsets of U that are subsets of an odd number of members of $\mathcal{F}$. Prove that $D(D(\mathcal{F})) = \mathcal{F}$.*

Proof. Let $f : 2^U \times 2^U \to \mathbb{Z}^*$ be $f(X,Y) = \#(\{Z \in 2^U \mid X \subseteq Z \wedge Z \subseteq Y\})$. Then

$$f(X,Y) = \begin{cases} 0 & \text{if } X \not\subseteq Y \\ 2^{\#(Y)-\#(X)} & \text{if } X \subseteq Y \end{cases}$$

, and $f(X,Y) \equiv_2 1$ iff $X = Y$, so $\sum_{F \in \mathcal{F}} f(X,F) \equiv_2 \mathbb{1}_{\mathcal{F}}(X)$.

Let $g : 2^U \times 2^{(2^U)} \to \mathbb{Z}^*$ be $g(X,\mathcal{F}) = \#(\{Y \in \mathcal{F} \mid X \subseteq Y\})$, so $D(\mathcal{F}) = \{X \in 2^U \mid g(X,\mathcal{F}) \equiv_2 1\}$. Then

$$\begin{aligned}
g(X, D(\mathcal{F})) &= \#(\{Y \in D(\mathcal{F}) \mid X \subseteq Y\}) \\
&= \#(\{Y \in 2^U \mid Y \in D(\mathcal{F}) \wedge X \subseteq Y\}) \\
&= \#(\{Y \in 2^U \mid g(Y,\mathcal{F}) \equiv_2 1 \wedge X \subseteq Y\}) \\
&\equiv_2 \sum_{X \subseteq Y \in 2^U} g(Y,\mathcal{F}) \\
&= \sum_{X \subseteq Y \in 2^U} \#(\{Z \in \mathcal{F} \mid Y \subseteq Z\}) \\
&= \sum_{X \subseteq Y \in 2^U} \sum_{Z \in \mathcal{F}} \mathbb{1}_{2^Z}(Y) \\
&= \sum_{Z \in \mathcal{F}} \sum_{X \subseteq Y \in 2^U} \mathbb{1}_{2^Z}(Y) \\
&= \sum_{Z \in \mathcal{F}} \#(\{Y \in 2^U \mid X \subseteq Y \wedge Y \subseteq Z\}) \\
&= \sum_{Z \in \mathcal{F}} f(X,Z) \equiv_2 \mathbb{1}_{\mathcal{F}}(X)
\end{aligned}$$

, so $D(D(\mathcal{F})) = \{X \in 2^U \mid g(X, D(\mathcal{F})) \equiv_2 1\} = \{X \in 2^U \mid X \in \mathcal{F}\} = \mathcal{F}$. □